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A NEW LOOK AT BERGSTRÖM'S THEOREM ON CONVERGENCE IN DISTRIBUTION FOR SUMS OF DEPENDENT RANDOM VARIABLES

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ABSTRACT
In his 1972 *Periodica Mathematica Hungarica* paper, H. Bergström stated a theorem on convergence in distribution for triangular arrays of dependent random variables satisfying a ϕ-mixing condition. A gap in his proof of this theorem is explained and a more general version is proved under weakened hypotheses. The method used consists of comparisons between the given array and associated arrays which are parameterized by a truncation variable. In addition to the main theorem, this method yields a proof of equality of limiting finite-dimensional distributions for processes generated by the given associated arrays as well as the result that if a limit distribution for the centered row sums does exist, it must be infinitely divisible. Several corollaries to the main theorem specialize this result for convergence to distributions within certain subclasses of the infinitely divisible laws.

1. Introduction

In the early 1970's, H. Bergström [1–3] wrote a series of papers concerning comparison methods for convergence in distribution of row sums of triangular arrays of dependent random variables. In [1] and [3], the basic idea was to approximate the row sums by suitable partial sums, with gaps determined by a partition of the row index, and then compare these partial sums with associated ones formed from independent summands. However, in [2] the comparison method was quite different and consisted of approximating the given array by an array in which each row is the sum of two independent vectors, the first one Gaussian and the second with independent components.

The major theorem of [2] states that for triangular arrays of dependent random variables, satisfying a ϕ-mixing condition with specified decay rate and certain other preliminary conditions, the joint fulfilment of three limit relations

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is both necessary and sufficient for the convergence in distribution of the row sums of such an array to an infinitely divisible law. These three relations are essentially the same as those in the classical case of triangular arrays having independence within each row (see [5]). A close examination of Bergström's paper reveals that the proof as given there will go through only in the event that each row of the array is part of a stationary sequence. Nevertheless, the theorem is true as stated and the proof can be corrected to demonstrate the validity of the result. In fact, not only is the theorem true but it can be improved, as can some of the key propositions leading up to it. Furthermore, the nature of the preliminary conditions can be clarified and, consequently, these conditions can be weakened.

2. Definitions and notation

We consider triangular arrays $\{X_{n,j}\}$, where for each positive integer n the random variables $X_{n,j}$, $j = 1, 2, \cdots, k(n)$, are defined on the same probability space and $\{k(n)\}$ is a non-decreasing unbounded sequence of positive integers. If $I(A)$ denotes the indicator function of the event A and $\varepsilon > 0$, we define

$$\hat{X}_{n,j}(\varepsilon) = X_{n,j} I(\{|X_{n,j}| \leqq \varepsilon\}),$$

$$\tilde{X}_{n,j}(\varepsilon) = X_{n,j} - \hat{X}_{n,j}(\varepsilon) \tag{2.1}$$

and

$$\mu_{n,j}(\varepsilon) = E(\hat{X}_{n,j}(\varepsilon)) = \int_{\{|X_{n,j}| \leqq \varepsilon\}} X_{n,j} dP.$$

$\{X_{n,j}\}$ satisfies the weak dependency condition known as ϕ-mixing if and only if the mixing coefficient $\phi(n, k)$, defined for $k = 1, 2, \cdots, k(n) - 1$, $n = 1, 2, 3, \cdots$, to be the supremum of $|P(A \cap B) - P(A)P(B)|/P(A)$ taken over all events A and B such that for some j, $1 \leqq j < j + k \leqq k(n)$, $A \in \mathcal{B}(X_{n,1}, \cdots, X_{n,j})$, $P(A) > 0$, and $B \in \mathcal{B}(X_{n,j+k}, \cdots, X_{n,k(n)})$, has the property that

$$\lim_{k\to\infty} \limsup_{n\to\infty} \phi(n, k) = 0. \tag{2.2}$$

This mixing coefficient satisfies the decay rate known as Ibragimov's condition [6] if and only if

$$\lim_{k\to\infty} \limsup_{n\to\infty} \sum_{j=k}^{k(n)-1} [\phi(n, j)]^{\frac{1}{2}} = 0. \tag{2.3}$$

We next consider conditions on truncated versions of $\{X_{n,j}\}$ which are related to the question of convergence in distribution, for suitable constants A_n, of the sums

$$S_n = \sum_{j=1}^{k(n)} X_{n,j} - A_n.$$

We list six such conditions here, although another appears in Section 7.

$uan(\varepsilon)$: $\lim_{n\to\infty} \max_{1\leq j\leq k(n)} P(\{|X_{n,j}| > \varepsilon\}) = 0.$

$stb(\varepsilon)$: $\limsup_{n\to\infty} \sum_{j=1}^{k(n)} P(\{|X_{n,j}| > \varepsilon\}) < \infty.$

$smb(\varepsilon)$: $\limsup_{n\to\infty} \sum_{j=1}^{k(n)} |\mu_{n,j}(\varepsilon)| < \infty.$

$ssb(\varepsilon)$: $\limsup_{n\to\infty} \sum_{j=1}^{k(n)} E([\hat{X}_{n,j}(\varepsilon)]^2) < \infty.$

$svb(\varepsilon)$: $\limsup_{n\to\infty} \sum_{j=1}^{k(n)} \operatorname{var}(\hat{X}_{n,j}(\varepsilon)) < \infty.$

$sjn(\varepsilon)$: $\lim_{n\to\infty} \sum_{j=1}^{k(n)-k} P(\{|X_{n,j}| > \varepsilon, |X_{n,j+k}| > \varepsilon\}) = 0$ for every positive integer k.

We say that $\{X_{n,j}\}$ is uniformly asymptotically negligible, *uan*, if and only if *uan* (ε) is satisfied for every $\varepsilon > 0$. Similarly, we can define: sums of tail probabilities bounded, *stb*; sums of absolute values of means bounded, *smb*; sums of second moments bounded, *ssb*; sums of variances bounded, *svb*; and sums of joint tail probabilities asymptotically negligible, *sjn*.

In the case that for each n, the random variables $X_{n,j}$, $j = 1, 2, \cdots, k(n)$, are independent, the basic assumption made in many theorems concerning convergence in distribution of the sums $\{S_n\}$ is that $\{X_{n,j}\}$ satisfies *uan*. If this condition is satisfied in the independent case and, in fact, the sums $\{S_n\}$ do converge in distribution, then it turns out that $\{X_{n,j}\}$ must satisfy *stb* and *svb*. Furthermore, in the case of independence, *uan* and *stb* imply that *sjn* is satisfied as well. In summary, it turns out that for independent *uan* random variables, *stb*, *svb*, and *sjn* are necessary conditions for convergence in distribution of the sums $\{S_n\}$.

If $e^{\psi(v)}$ is the characteristic function of an infinitely divisible distribution, then

$$\psi(v) = iv\gamma - v^2\sigma^2/2 + \int_{\{|u|>0\}} [e^{iuv} - 1 - iuv/(1+u^2)]dQ(u), \tag{2.4}$$

where $-\infty < \gamma < \infty$, $\sigma^2 \geqq 0$, and Q is non-decreasing on $(-\infty, 0)$ and $(0, \infty)$ with $\lim_{|u|\to\infty} Q(u) = 0$, and $\int_{\{0<|u|\leqq\varepsilon\}} u^2 dQ(u) < \infty$ for every $\varepsilon > 0$. Thus for any $\varepsilon > 0$. we can write

$$\psi(v) = iv\gamma(\varepsilon) - v^2\sigma^2/2 + \int_{\{|u|>\varepsilon\}} (e^{iuv} - 1)dQ(u) + \int_{\{0<|u|\leqq\varepsilon\}} (e^{iuv} - 1 - iuv)dQ(u), \tag{2.5}$$

where

$$\gamma(\varepsilon) = \gamma + \int_{\{0<|u|\leqq\varepsilon\}} u^3/(1+u^2)dQ(u) - \int_{\{|u|>\varepsilon\}} u/(1+u^2)dQ(u). \tag{2.6}$$

Note that if Q is known and $\gamma(\varepsilon_0)$ is known for some $\varepsilon_0 > 0$, then γ is known, as is $\gamma(\varepsilon)$ for all $\varepsilon > 0$. In fact we have, for $0 < \varepsilon < \varepsilon_0$,

$$\gamma(\varepsilon_0) - \gamma(\varepsilon) = \int_{\{\varepsilon<|u|\leqq\varepsilon_0\}} u dQ(u). \tag{2.7}$$

3. Statements of results and methods of proof

MAIN THEOREM. *Assume $\{X_{n,j}\}$ satisfies the preliminary conditions: ϕ-mixing with coefficient fulfilling Ibragimov's condition, uan, stb, svb, and sjn. Then*

$$S_n = \sum_{j=1}^{k(n)} X_{n,j} - A_n$$

converges in distribution as $n \to \infty$ to the infinitely divisible law determined in the form (2.4) by Q, γ, and σ^2 if and only if

(i) $$\lim_{n\to\infty} \sum_{j=1}^{k(n)} P(\{X_{n,j} > u\}) = -Q(u)$$

whenever $u > 0$ is a continuity point of Q, and

$$\lim_{n\to\infty} \sum_{j=1}^{k(n)} P(\{X_{n,j} < u\}) = Q(u)$$

whenever $u < 0$ is a continuity point of Q;

(ii) *for some $\varepsilon_0 > 0$ such that $\pm\varepsilon_0$ are continuity points of Q,*

$$\lim_{n\to\infty} \left[\sum_{j=1}^{k(n)} \mu_{n,j}(\varepsilon_0) - A_n \right] = \gamma(\varepsilon_0);$$

and

(iii) $\lim_{\varepsilon \to 0} \liminf_{n \to \infty} \operatorname{var}\left(\sum_{j=1}^{k(n)} \hat{X}_{n,j}(\varepsilon)\right) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \operatorname{var}\left(\sum_{j=1}^{k(n)} \hat{X}_{n,j}(\varepsilon)\right) = \sigma^2$,

when $\varepsilon \to 0$ *through positive real numbers such that* $\pm \varepsilon$ *are continuity points of* Q.

Observe that this theorem includes the classical result for independent *uan* random variables, which states that (i), (ii), and (iii) are necessary and sufficient for convergence in distribution of $\{S_n\}$ to the infinitely divisible law determined by Q, γ, and σ^2, without further preliminary conditions. We have already mentioned the necessity of *stb*, *svb*, and *sjn* in the independent *uan* case and for sufficiency it is enough to realize that (i) implies *stb*, (iii) and independence give *svb*, and, as before, independence, *uan*, and *stb* yield *sjn*. In Bergström's theorem [2] only the case $A_n = 0$ is considered, the preliminary condition *svb* is replaced by the stronger condition *ssb*, and the additional condition *smb* is imposed. Bergström also states that $\sigma^2 > 0$, which eliminates the stable laws as possible limits, but this restriction is certainly not needed.

In order to prove the theorem, we shall use the classical results in combination with a modified version of the comparison method of Bergström. The comparison involves the given ϕ-mixing array, centered at the truncated mean $\mu_{n,j}(\varepsilon_0)$, and associated arrays parameterized by ε, for $0 < \varepsilon \leqq \varepsilon_0$. These arrays are constructed as follows: given $0 < \varepsilon \leqq \varepsilon_0$ and a positive integer n,

(a) define $\hat{Y}_{n,j}(\varepsilon_0, \varepsilon)$, $j = 1, 2, \cdots, k(n)$, to be a Gaussian random vector, independent of the $X_{n,j}$ and having the same mean vector and covariance matrix as

$$[X_{n,j} - \mu_{n,j}(\varepsilon_0)] I(\{|X_{n,j} - \mu_{n,j}(\varepsilon_0)| \leqq \varepsilon\}), \qquad j = 1, 2, \cdots, k(n);$$

(b) let $\tilde{Y}_{n,j}(\varepsilon_0, \varepsilon)$, $j = 1, 2, \cdots, k(n)$, be independent random variables, independent of the $\hat{Y}_{n,j}(\varepsilon_0, \varepsilon)$ and of the $X_{n,j}$ as well, such that $\tilde{Y}_{n,j}(\varepsilon_0, \varepsilon)$ has the same distribution as

$$[X_{n,j} - \mu_{n,j}(\varepsilon_0)] I(\{|X_{n,j} - \mu_{n,j}(\varepsilon_0)| > \varepsilon\}), \qquad j = 1, 2, \cdots, k(n);$$

(c) finally, let $Y_{n,j}(\varepsilon_0, \varepsilon) = \hat{Y}_{n,j}(\varepsilon_0, \varepsilon) + \tilde{Y}_{n,j}(\varepsilon_0, \varepsilon)$, $j = 1, 2, \cdots, k(n)$.

Our modifications to Bergström's method can now be described. First of all, we stick to Fourier analysis, i.e. characteristic functions, rather than using his Gaussian transform. Secondly, we define the Gaussian parts of our associated arrays without the changes in the covariance matrices that cause Bergström's proof to break down (see the end of Section 4). Thirdly, we compensate for use of these unchanged covariance matrices by a slightly more delicate analysis of the comparison between the given and the associated arrays.

We can now state the propositions which combine to prove the main theorem. Some of these go beyond the minimum needed for this theorem and hence have interest in their own right. This is particularly true of the fundamental comparison proposition, Proposition A, which leads to equality of limiting finite-dimensional distributions for processes generated by the given and associated arrays. In addition, Proposition D shows that under the preliminary assumptions of the main theorem, the only possible limit laws for the sums $\{S_n\}$ are, as in the independent case, the infinitely divisible distributions.

PROPOSITION A. *Let* $\{X_{n,j}\}$ *satisfy the preliminary conditions of the main theorem. For arbitrary* $\varepsilon_0 > 0$, *let* $Z_{n,j} = X_{n,j} - \mu_{n,j}(\varepsilon_0)$ *and write* $Y_{n,j}(\varepsilon)$ *for* $Y_{n,j}(\varepsilon_0, \varepsilon)$, $0 < \varepsilon \leqq \varepsilon_0$. *Let* $v = \{v_{n,j}\}$ *denote an array of real constants and define*

$$\| v \| = \sup\{| v_{n,j} | : j = 1, 2, \cdots, k(n), n = 1, 2, 3, \cdots\}.$$

Finally, define

$$S(v, n, \varepsilon) = E\left(\exp\left[i \sum_{j=1}^{k(n)} v_{n,j} Z_{n,j}\right]\right) - E\left(\exp\left[i \sum_{j=1}^{k(n)} v_{n,j} Y_{n,j}(\varepsilon)\right]\right). \tag{3.1}$$

Then for every $V > 0$,

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \sup_{\|v\| \leqq V} | S(v, n, \varepsilon) | = 0.$$

COROLLARY 1. *For* $n = 1, 2, 3, \cdots$, *let* θ_n *be a continuous, strictly increasing mapping of the interval* $[0, 1]$ *onto itself. Define the processes* $U_n(t)$ *and* $T_n(t, \varepsilon)$ *for* $0 \leqq t \leqq 1$ *by*

$$U_n(t) = \sum_{j=1}^{[k(n)\theta_n(t)]} Z_{n,j} \quad \text{and} \quad T_n(t, \varepsilon) = \sum_{j=1}^{[k(n)\theta_n(t)]} Y_{n,j}(\varepsilon), \tag{3.2}$$

where, in this corollary (and its proof), $[x]$ *denotes the greatest integer not exceeding* x.

Then the finite dimensional distributions of $\{U_n(t) : 0 \leqq t \leqq 1\}$ *converge as* $n \to \infty$ *if and only if the finite dimensional distributions of* $\{T_n(t, \varepsilon) : 0 \leqq t \leqq 1\}$ *converge when* $n \to \infty$ *followed by* $\varepsilon \to 0$. *In this case, the limiting distributions are the same.*

COROLLARY 2. $\sum_{j=1}^{k(n)} Z_{n,j}$ *converges in distribution as* $n \to \infty$ *if and only if* $\sum_{j=1}^{k(n)} Y_{n,j}(\varepsilon)$ *converges in distribution as* $n \to \infty$ *and then* $\varepsilon \to 0$. *In this case, the limiting distributions are the same.*

COROLLARY 3. $S_n = \sum_{j=1}^{k(n)} X_{n,j} - A_n$ *converges in distribution as* $n \to \infty$ *if and* *only if* $\sum_{j=1}^{k(n)} Y_{n,j}(\varepsilon)+\sum_{j=1}^{k(n)} \mu_{n,j}(\varepsilon_0)-A_n$ *converges in distribution as* $n\to\infty$ *and then* $\varepsilon\to 0$. *In this case, the limiting distributions are the same.*

PROPOSITION B. *Let* $\{X_{n,j}\}$ *satisfy uan and condition* (i) *of the main theorem. Let* $\varepsilon_0>0$ *with* $\pm\varepsilon_0$ *continuity points of the function* Q *specified in* (i). *Choose* $0<\varepsilon\leqq\varepsilon_0$ *so that* $\pm\varepsilon$ *are continuity points of* Q *and define* $\tilde{W}_{n,j}(\varepsilon)$, $j=1,2,\cdots,k(n)$, *to be independent random variables, with* $\tilde{W}_{n,j}(\varepsilon)$ *distributed the same as* $\tilde{X}_{n,j}(\varepsilon)$. *Then* $\sum_{j=1}^{k(n)}[\tilde{W}_{n,j}(\varepsilon)-\mu_{n,j}(\varepsilon_0)+\mu_{n,j}(\varepsilon)]$ *converges in distribution as* $n\to\infty$ *to the infinitely divisible law determined in the form* (2.5) *by* Q_ε, $\gamma_\varepsilon(\varepsilon_0)=0$, $\sigma_\varepsilon^2=0$, *where*

$$Q_\varepsilon(u)=\begin{cases}Q(u), & |u|>\varepsilon,\\ Q(\varepsilon), & 0<u\leqq\varepsilon,\\ Q(-\varepsilon), & -\varepsilon\leqq u<0.\end{cases}\tag{3.3}$$

COROLLARY. *Let* $\{X_{n,j}\}$, Q, ε_0, *and* ε *be as in Proposition B. Write* $\tilde{Y}_{n,j}(\varepsilon)$ *for* $\tilde{Y}_{n,j}(\varepsilon_0,\varepsilon)$ *and let* $\nu_{n,j}(\varepsilon)=E(\hat{Y}_{n,j}(\varepsilon_0,\varepsilon))$. *Then* $\sum_{j=1}^{k(n)}[\tilde{Y}_{n,j}(\varepsilon)+\nu_{n,j}(\varepsilon)]$ *converges in distribution as* $n\to\infty$ *to the infinitely divisible law determined in the form* (2.5) *by* Q_ε, $\gamma_\varepsilon(\varepsilon_0)=0$, *and* $\sigma_\varepsilon^2=0$.

PROPOSITION C. *Let* $\{X_{n,j}\}$, Q, ε_0, *and* ε *be as in Proposition B and, in addition, assume that* $\{Z_{n,j}\}=\{X_{n,j}-\mu_{n,j}(\varepsilon_0)\}$ *satisfies condition* (iii) *of the main theorem. Then* $\sum_{j=1}^{k(n)} Y_{n,j}(\varepsilon)$ *converges in distribution, when* $n\to\infty$ *followed by* $\varepsilon\to 0$ *through positive values such that* $\pm\varepsilon$ *are continuity points of* Q, *to the infinitely divisible law determined in the form* (2.5) *by* Q, $\gamma_0(\varepsilon_0)=0$, *and* σ^2.

COROLLARY. *Assume the hypotheses of Proposition C with an* $\varepsilon_0>0$ *such that condition* (ii) *of the main theorem holds. Then* $\sum_{j=1}^{k(n)} Y_{n,j}(\varepsilon)+\sum_{j=1}^{k(n)}\mu_{n,j}(\varepsilon_0)-A_n$ *converges in distribution, when* $n\to\infty$ *followed by* $\varepsilon\to 0$ *through positive values such that* $\pm\varepsilon$ *are continuity points of* Q, *to the infinitely divisible law determined in the form* (2.5) *by* Q, $\gamma(\varepsilon_0)$, *and* σ^2.

PROPOSITION D. *Suppose* $\{X_{n,j}\}$ *satisfies the preliminary conditions of the main theorem and*

$$S_n=\sum_{j=1}^{k(n)} X_{n,j}-A_n$$

converges in distribution as $n\to\infty$. *Then the limit distribution is infinitely divisible. Furthermore, if this distribution is determined in the form* (2.4) *by* Q, γ, *and* σ^2, *then conditions* (i), (ii), *and* (iii) *of the main theorem are satisfied, with* (ii) *holding for every* $\varepsilon_0>0$ *such that* $\pm\varepsilon_0$ *are continuity points of* Q.

Note that Proposition D gives the necessity part of the main theorem while the corollary to Proposition C, Lemma 17 of Section 4, and Corollary 3 to Proposition A yield the sufficiency. In connection with the assumed limit law necessarily being infinitely divisible, see Bergström [3].

4. Preliminary lemmas and counterexamples

We first consider the effect of centering on our various truncation conditions. We let $\{\alpha_{n,j}\}$ be a triangular array of real constants such that

$$\lim_{n\to\infty} \max_{1\leq j\leq k(n)} |\alpha_{n,j}| = 0.$$

The following results are immediate.

LEMMA 1. *If*

$$\limsup_{n\to\infty} \sum_{j=1}^{k(n)} |\alpha_{n,j}| < \infty$$

and $\{X_{n,j}\}$ *satisfies stb, then* $\{X_{n,j}\}$ *satisfies smb if and only if* $\{X_{n,j}-\alpha_{n,j}\}$ *does.*

LEMMA 2. *Assume* $\{X_{n,j}\}$ *satisfies stb. Then* $\{X_{n,j}\}$ *satisfies smb* (ε_0) *for some* $\varepsilon_0>0$ *if and only if* $\{X_{n,j}\}$ *satisfies smb. The same holds for ssb* (ε_0) *and ssb as well as svb* (ε_0) *and svb.*

LEMMA 3. *Assume* $\{X_{n,j}\}$ *satisfies uan. Then for every* $r>0$ *and* $\varepsilon>0$,

$$\max_{1\leq j\leq k(n)} E(|\hat{X}_{n,j}(\varepsilon)|^r)\to 0 \qquad \text{as } n\to\infty.$$

In particular, $|\mu_{n,j}(\varepsilon)|\leqq E(|\hat{X}_{n,j}(\varepsilon)|)$, *so that* $\max_{1\leq j\leq k(n)}|\mu_{n,j}(\varepsilon)|\to 0$ *as* $n\to\infty$ *for every* $\varepsilon>0$.

LEMMA 4. *Let* $\{X_{n,j}\}$ *satisfy uan and stb. Then for every* $\varepsilon_0>0$, $\{X_{n,j}-\mu_{n,j}(\varepsilon_0)\}$ *satisfies smb.*

LEMMA 5. *Suppose* $\{X_{n,j}\}$ *satisfies uan and stb. For* $0<\varepsilon\leqq\varepsilon_0$, *let*

$$\nu_{n,j}(\varepsilon)=\int_{\{|X_{n,j}-\mu_{n,j}(\varepsilon_0)|\leq\varepsilon\}}[X_{n,j}-\mu_{n,j}(\varepsilon_0)]dP.$$

Then $\max_{1\leq j\leq k(n)}|\nu_{n,j}(\varepsilon)|\to 0$ *as* $n\to\infty$,

$$\limsup_{n\to\infty}\sum_{j=1}^{k(n)}|\nu_{n,j}(\varepsilon)|<\infty,$$

and hence $\sum_{j=1}^{k(n)}[\nu_{n,j}(\varepsilon)]^2\to 0$ *as* $n\to\infty$. *Consequently, if we write*

$$\hat{Z}_{n,j}(\varepsilon) = [X_{n,j} - \mu_{n,j}(\varepsilon_0)]I(\{|X_{n,j} - \mu_{n,j}(\varepsilon_0)| \leqq \varepsilon\}),$$

$$0 \leqq \sum_{j=1}^{k(n)} [E([\hat{Z}_{n,j}(\varepsilon)]^2) - \operatorname{var}(\hat{Z}_{n,j}(\varepsilon))] = \sum_{j=1}^{k(n)} [\nu_{n,j}(\varepsilon)]^2 \to 0 \qquad \text{as } n \to \infty.$$

LEMMA 6. *Let $\{X_{n,j}\}$ satisfy uan and stb. Then $\{X_{n,j}\}$ satisfies svb if and only if $\{X_{n,j} - \mu_{n,j}(\varepsilon_0)\}$ satisfies ssb.*

We now examine the necessity of the truncation conditions in the independent *uan* case.

LEMMA 7. *Assume $\{X_{n,j}\}$ satisfies uan; for each n, $X_{n,1}, X_{n,2}, \cdots, X_{n,k(n)}$ are independent; and there exist constants $\{A_n\}$ such that the sums*

$$S_n = \sum_{j=1}^{k(n)} X_{n,j} - A_n$$

converge in distribution as $n \to \infty$. Then according to the classical results (see Gnedenko and Kolmogorov [5]), *for every $\varepsilon_0 > 0$ such that $\pm\varepsilon_0$ are continuity points of the function Q associated with the limiting infinitely divisible law,*

$$\gamma_n(\varepsilon_0) = \sum_{j=1}^{k(n)} \mu_{n,j}(\varepsilon_0) - A_n$$

converges as $n \to \infty$, which is equivalent to saying that $\sum_{j=1}^{k(n)} [X_{n,j} - \mu_{n,j}(\varepsilon_0)]$ converges in distribution as $n \to \infty$. Thus, again by the classical results, we see that $\{X_{n,j} - \mu_{n,j}(\varepsilon_0)\}$ satisfies stb and ssb. But this is equivalent to $\{X_{n,j}\}$ satisfying stb and svb. Furthermore, $\{X_{n,j}\}$ satisfying uan and stb implies $\{X_{n,j} - \mu_{n,j}(\varepsilon_0)\}$ satisfies smb and when independence is considered we also get that both $\{X_{n,j}\}$ and $\{X_{n,j} - \mu_{n,j}(\varepsilon_0)\}$ satisfy sjn.

EXAMPLE 1. For a counterexample to the necessity of *smb* and *ssb* for $\{X_{n,j}\}$, we suppose that for every positive integer n, $X_{n,1}, X_{n,2}, \cdots, X_{n,n}$ are independent and normally distributed random variables with variance n^{-1} and mean

$$\mu_{n,j} = E(X_{n,j}) = [\log(n+1)]^{-\frac{1}{4}}(-1)^j j^{-\frac{1}{2}},$$

for $j = 1, 2, \cdots, n$. Note that

$$\max_{1 \leqq j \leqq n} |\mu_{n,j}| = [\log(n+1)]^{-\frac{1}{4}} \to 0 \qquad \text{as } n \to \infty.$$

By Chebyshev's inequality, $P(\{|X_{n,j} - \mu_{n,j}| > \varepsilon\}) \leqq \varepsilon^{-2} n^{-1}$, $j = 1, 2, \cdots, n$, and hence $\{X_{n,j}\}$ satisfies *uan* and *stb*. Since $\sum_{j=1}^{n} [X_{n,j} - \mu_{n,j}]$ has an $N(0,1)$ distribution and

$$\sum_{j=1}^{n} \mu_{n,j} = [\log(n+1)]^{-\frac{1}{4}} \sum_{j=1}^{n} (-1)^j j^{-\frac{1}{2}} \to 0 \qquad \text{as } n \to \infty,$$

we see that $\sum_{j=1}^n X_{n,j}$ converges in distribution to $N(0,1)$ as $n \to \infty$. It is easy to show that $\sum_{j=1}^n E([\hat{X}_{n,j}(\varepsilon)]^2) \to \infty$ as $n \to \infty$ and $\sum_{j=1}^n |\mu_{n,j}(\varepsilon)| \to \infty$ as $n \to \infty$, even though $\sum_{j=1}^n \mu_{n,j}(\varepsilon) \to 0$ as $n \to \infty$. Thus we see that the convergence of $\sum_{j=1}^n X_{n,j}$ is equivalent to the convergence of $\sum_{j=1}^n [X_{n,j} - \mu_{n,j}(\varepsilon)]$, with even the same limit distribution, and yet the conditions *smb* and *ssb* are not satisfied for $\{X_{n,j}\}$. One last thing to observe is that despite the fact that $\sum_{j=1}^n E([\hat{X}_{n,j}(\varepsilon)]^2) \to \infty$ as $n \to \infty$ and $\sum_{j=1}^n [\mu_{n,j}(\varepsilon)]^2 \to \infty$ as $n \to \infty$, we still have $\sum_{j=1}^n \operatorname{var}(\hat{X}_{n,j}(\varepsilon)) \to 1$ as $n \to \infty$.

We next want to consider the effect of condition (i) of the main theorem, a condition which implies *stb*.

LEMMA 8. *If* $\max_{1 \leq j \leq k(n)} |\alpha_{n,j}| \to 0$ *as* $n \to \infty$, *then* $\{X_{n,j}\}$ *satisfies* (i) *for a given function* Q *if and only if* $\{X_{n,j} - \alpha_{n,j}\}$ *satisfies* (i) *for the same* Q.

LEMMA 9. *Let* $\{X_{n,j}\}$ *satisfy uan and condition* (i). *Use the notation of Lemma* 5 *and assume that* $\pm\varepsilon$ *are continuity points of* Q. *Then*

$$\sum_{j=1}^{k(n)} |\nu_{n,j}(\varepsilon) + [\mu_{n,j}(\varepsilon_0) - \mu_{n,j}(\varepsilon)]| \to 0 \qquad \text{as } n \to \infty.$$

If $\pm\varepsilon_0$ *are also continuity points of* Q, *then for* $0 < \varepsilon < \varepsilon_0$,

$$\sum_{j=1}^{k(n)} \nu_{n,j}(\varepsilon) \to -\int_{\{\varepsilon < |u| \leq \varepsilon_0\}} u\,dQ(u) \qquad \text{as } n \to \infty$$

and

$$\sum_{j=1}^{k(n)} |\nu_{n,j}(\varepsilon_0)| \to 0 \qquad \text{as } n \to \infty.$$

LEMMA 10. *Let* $\{X_{n,j}\}$ *satisfy uan and condition* (i). *Again use the terminology of Lemma* 5 *with the assumption that* $\pm\varepsilon$ *are continuity points of* Q. *Then*

$$\sum_{j=1}^{k(n)} |E([\hat{Z}_{n,j}(\varepsilon)]^2) - \operatorname{var}(\hat{X}_{n,j}(\varepsilon))| \to 0 \qquad \text{as } n \to \infty$$

and hence

$$\sum_{j=1}^{k(n)} |\operatorname{var}(\hat{Z}_{n,j}(\varepsilon)) - \operatorname{var}(\hat{X}_{n,j}(\varepsilon))| \to \infty \qquad \text{as } n \to \infty.$$

LEMMA 11. *If* $\{X_{n,j}\}$ *satisfies stb and sjn, then for every* $r > 0$, *every positive integer* k, *and every* $0 < \varepsilon \leqq \varepsilon_0$,

$$\lim_{n\to\infty}\sum_{j=1}^{k(n)-k}\int_{\{|X_{n,j}|>\varepsilon\}}|\hat{X}_{n,j+k}(\varepsilon_0)|^r dP=0$$

and

$$\lim_{n\to\infty}\sum_{j=1}^{k(n)-k}\int_{\{|X_{n,j+k}|>\varepsilon\}}|\hat{X}_{n,j}(\varepsilon_0)|^r dP=0.$$

Next, by applying Hölder's inequality, first for expectations and then for sums, we see that if k and s are non-negative integers, r_1, r_2, and r_3 are non-negative real numbers, and $r=r_1+r_2+r_3$, then for every $\varepsilon>0$

$$\sum_{j=1}^{k(n)-k-s}E(|\hat{X}_{n,j}(\varepsilon)|^{r_1}|\hat{X}_{n,j+k}(\varepsilon)|^{r_2}|\hat{X}_{n,j+k+s}(\varepsilon)|^{r_3})\leqq\sum_{j=1}^{k(n)}E(|\hat{X}_{n,j}(\varepsilon)|^r). \tag{4.1}$$

Furthermore, if $\{X_{n,j}\}$ satisfies *ssb*, then

$$K_2(\varepsilon)=\limsup_{n\to\infty}\sum_{j=1}^{k(n)}E([\hat{X}_{n,j}(\varepsilon)]^2)$$

is non-increasing as ε decreases. Hence if we also assume $r>2$, then

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\sum_{j=1}^{k(n)}E(|\hat{X}_{n,j}(\varepsilon)|^r)\leqq\lim_{\varepsilon\to 0}\varepsilon^{r-2}K_2(\varepsilon)=0. \tag{4.2}$$

LEMMA 12. *Assume $\{X_{n,j}\}$ satisfies stb, ssb, and sjn. Let k and s be non-negative integers, at least one of which is positive; let r_1, r_2 and r_3 be non-negative real numbers, at least two of which are positive, with $r=r_1+r_2+r_3$; and suppose that either* (a) $r>2$ *or* (b) $r=2$ *and $K_2(\varepsilon')\to 0$ as $\varepsilon'\to 0$. Then for every $\varepsilon>0$,*

$$\lim_{n\to\infty}\sum_{j=1}^{k(n)-k-s}E(|\hat{X}_{n,j}(\varepsilon)|^{r_1}|\hat{X}_{n,j+k}(\varepsilon)|^{r_2}|\hat{X}_{n,j+k+s}(\varepsilon)|^{r_3}=0.$$

The next result, which is a direct extension of Lemma 5, involves covariances and product moments.

LEMMA 13. *Suppose $\{X_{n,j}\}$ satisfies uan and stb. Then for every positive integer k and $0<\varepsilon\leqq\varepsilon_0$,*

$$\sum_{j=1}^{k(n)-k}|E(\hat{Z}_{n,j}(\varepsilon)\hat{Z}_{n,j+k}(\varepsilon))-\operatorname{cov}(\hat{Z}_{n,j}(\varepsilon),\hat{Z}_{n,j+k}(\varepsilon))|$$

$$=\sum_{j=1}^{k(n)-k}|\nu_{n,j}(\varepsilon)\nu_{n,j+k}(\varepsilon)|\to 0\qquad \text{as } n\to\infty.$$

LEMMA 14. *Let $\{X_{n,j}\}$ satisfy uan and stb. Then for every positive integer k and $0<\varepsilon\leqq\varepsilon_0$,*

$$\limsup_{n\to\infty} \sum_{j=1}^{k(n)-k} |E(\hat{Z}_{n,j}(\varepsilon)\hat{Z}_{n,j+k}(\varepsilon)) - \operatorname{cov}(\hat{X}_{n,j}(\varepsilon), \hat{X}_{n,j+k}(\varepsilon))| < \infty.$$

LEMMA 15. *Let $\{X_{n,j}\}$ satisfy uan and condition* (i). *Assume $\pm\varepsilon$ are continuity points of Q. Then for every positive integer k,*

$$\sum_{j=1}^{k(n)-k} |E(\hat{Z}_{n,j}(\varepsilon)\hat{Z}_{n,j+k}(\varepsilon)) - \operatorname{cov}(\hat{X}_{n,j}(\varepsilon), \hat{X}_{n,j+k}(\varepsilon))| \to 0 \quad \text{as } n \to \infty$$

and hence

$$\sum_{j=1}^{k(n)-k} |\operatorname{cov}(\hat{Z}_{n,j}(\varepsilon), \hat{Z}_{n,j+k}(\varepsilon)) - \operatorname{cov}(\hat{X}_{n,j}(\varepsilon), \hat{X}_{n,j+k}(\varepsilon))| \to 0 \quad \text{as } n \to \infty.$$

Finally, let's consider what happens if $\{X_{n,j}\}$ is ϕ-mixing. We need the following basic result; see Billingsley [4]. If $1 \leqq p \leqq \infty$, $p^{-1} + q^{-1} = 1$, and for some j, $1 \leqq j < j + k \leqq k(n)$, we have Y is measurable with respect to $\mathcal{B}(X_{n,1}, \cdots, X_{n,j})$, Z is measurable with respect to $\mathcal{B}(X_{n,j+k}, \cdots, X_{n,k(n)})$, $\| Y \|_p < \infty$, and $\| Z \|_q < \infty$, then

$$|E(YZ) - E(Y)E(Z)| \leqq 2[\phi(n,k)]^{1/p} \| Y \|_p \| Z \|_q. \tag{4.3}$$

Note that $\| X \|_r = [E(|X|^r)]^{1/r}$ for $1 \leqq r < \infty$, with $\| X \|_\infty = \operatorname{ess\,sup} |X|$. Furthermore, (4.3) applies only when Y and Z are both real valued. If one of these functions is complex valued, the inequality holds with 2 replaced by 4. If both are complex valued, the inequality holds with the factor 2 replaced by 8.

Now suppose that for $j = 1, 2, \cdots, k(n)$, $Z_{n,j}$ is measurable with respect to $\mathcal{B}(X_{n,j})$, i.e. $Z_{n,j} = g_{n,j}(X_{n,j})$ for some Borel function $g_{n,j}$, and $E([Z_{n,j}]^2) < \infty$. Then by (4.3) and the Cauchy–Schwarz inequality, for $0 \leqq K \leqq k(n) - 2$,

$$\begin{aligned} &\left| \operatorname{var}\left(\sum_{j=1}^{k(n)} Z_{n,j}\right) - \sum_{j=1}^{k(n)} \operatorname{var}(Z_{n,j}) - 2\sum_{k=1}^{K} \sum_{j=1}^{k(n)-k} \operatorname{cov}(Z_{n,j}, Z_{n,j+k}) \right| \\ &= \left| 2 \sum_{k=K+1}^{k(n)-1} \sum_{j=1}^{k(n)-k} \operatorname{cov}(Z_{n,j}, Z_{n,j+k}) \right| \\ &\leqq 4 \left(\sum_{k=K+1}^{k(n)-1} [\phi(n,k)]^{\frac{1}{2}} \right) \sum_{j=1}^{k(n)} \operatorname{var}(Z_{n,j}). \end{aligned} \tag{4.4}$$

LEMMA 16. *Suppose $\{X_{n,j}\}$ is ϕ-mixing with coefficient satisfying Ibragimov's condition. Assume that, for $j = 1, 2, \cdots, k(n)$, $n = 1, 2, 3, \cdots$, $Z_{n,j}$ is measurable with respect to $\mathcal{B}(X_{n,j})$ and $E([Z_{n,j}]^2) < \infty$. Then*

$$\limsup_{n\to\infty} \sum_{j=1}^{k(n)} \operatorname{var}(Z_{n,j}) < \infty \quad \textit{implies} \quad \limsup_{n\to\infty} \operatorname{var}\left(\sum_{j=1}^{k(n)} Z_{n,j}\right) < \infty$$

and

$$\lim_{n\to\infty}\sum_{j=1}^{k(n)}\operatorname{var}(Z_{n,j})=0 \quad \textit{implies} \quad \lim_{n\to\infty}\operatorname{var}\Big(\sum_{j=1}^{k(n)}Z_{n,j}\Big)=0.$$

Using Lemma 10, Lemma 15, and (4.4) we obtain our last result.

LEMMA 17. *Suppose $\{X_{n,j}\}$ is ϕ-mixing, with coefficient fulfilling Ibragimov's condition, and satisfies uan, svb, and condition* (i). *Then for $0<\varepsilon\leqq\varepsilon_0$ where $\pm\varepsilon$ are continuity points of Q, we have*

$$\Big|\operatorname{var}\Big(\sum_{j=1}^{k(n)}\hat{Z}_{n,j}(\varepsilon)\Big)-\operatorname{var}\Big(\sum_{j=1}^{k(n)}\hat{X}_{n,j}(\varepsilon)\Big)\Big|\to 0 \qquad \textit{as } n\to\infty.$$

Consequently, for any $\varepsilon_0>0$, $\{X_{n,j}\}$ satisfies condition (iii) *for $\sigma^2\geqq 0$ if and only if $\{X_{n,j}-\mu_{n,j}(\varepsilon_0)\}$ satisfies condition* (iii) *for the same σ^2.*

We conclude this section by examining the gap in Bergström's proof. The problem arises when he attempts to construct a Gaussian random vector $\hat{Y}_{n,j}(K,\varepsilon)$, $j=1,2,\cdots,k(n)$, with mean vector the same as $\hat{X}_{n,j}(\varepsilon)$, $j=1,2,\cdots,k(n)$, and with

$$(4.5)\qquad E(\hat{Y}_{n,j}(K,\varepsilon)\hat{Y}_{n,j-k}(K,\varepsilon))$$
$$=\begin{cases} E(\hat{X}_{n,j}(\varepsilon)\hat{X}_{n,j+k}(\varepsilon)), & 1\leqq k\leqq K,\quad j+k\leqq k(n),\\ E(\hat{X}_{n,j}(\varepsilon)\hat{X}_{n,j-k}(\varepsilon)), & \text{other } j \text{ and } k,\quad 0\leqq k<j\leqq k(n).\end{cases}$$

But, in fact, the supposed covariance matrix arising from this procedure need not be non-negative definite and hence the $\hat{Y}_{n,j}(K,\varepsilon)$ need not exist.

EXAMPLE 2. Let $k(n)=n\geqq 3$. Let $\{X_j\}$ be a sequence of independent random variables, each taking the two values ± 1 with equal probability. Let (Y,Z) be a random vector, independent of the $\{X_j\}$, each component of which has the same distribution as an X_j, but with $P(\{Y=1,Z=1\})$ close to 1/2, say 7/16. Let $X_{n,j}=n^{-\frac{1}{2}}X_j$ for $j=1,2,\cdots,n-2$. $X_{n,n-1}=n^{-\frac{1}{2}}Y$, and $X_{n,n}=n^{-\frac{1}{2}}Z$. Given $\varepsilon>0$, we assume that $n^{\frac{1}{2}}\varepsilon\geqq 1$, so that $\hat{X}_{n,j}(\varepsilon)=X_{n,j}$, $j=1,2,\cdots,n$. Then the means of the $\hat{X}_{n,j}(\varepsilon)$ are all zero and the covariance matrix has for its lower right-hand 3×3 submatrix

$$n^{-1}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 3/4\\ 0 & 3/4 & 1\end{bmatrix}$$

with the remainder of the matrix consisting of zeros except for n^{-1} on the main diagonal.

Now suppose $K=1$ and consider the covariance matrix proposed for $\hat{Y}_{n,j}(1,\varepsilon)$. The only change will be in the first off-diagonals, where all entries except the lowest are obtained from our original covariance matrix by shifting the entries up by one position and the lowest remains as it was. In other words, the proposed covariance matrix will have for its lower right hand 3×3 submatrix

$$n^{-1}\begin{bmatrix} 1 & 3/4 & 0 \\ 3/4 & 1 & 3/4 \\ 0 & 3/4 & 1 \end{bmatrix}$$

with the remainder as before, namely all zeros except for n^{-1} on the main diagonal. But this matrix is clearly not non-negative definite and hence cannot be a covariance matrix.

5. Proofs of Proposition A and its corollaries

First note that since $\{X_{n,j}\}$ is assumed to satisfy the preliminary conditions of the main theorem, $\{Z_{n,j}\}$ satisfies *uan*, *stb*, *smb*, *ssb*, and *sjn*, as well as being ϕ-mixing with coefficient fulfilling Ibragimov's condition. We define

$$\Phi_j(v,n,\varepsilon)=E\left(\exp\left[i\sum_{m=1}^{j}v_{n,m}Y_{n,m}(\varepsilon)+\sum_{m=j+1}^{k(n)}v_{n,m}Z_{n,m}\right]\right)$$

for $j=0,1,\cdots,k(n)$, with

$$\Delta_j(v,n,\varepsilon)=\Phi_{j-1}(v,n,\varepsilon)-\Phi_j(v,n,\varepsilon)$$

for $j=1,2,\cdots,k(n)$. Thus we want to prove that, for each $V>0$,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\sup_{\|v\|\leqq V}|S(v,n,\varepsilon)|=0, \tag{5.1}$$

where

$$\begin{aligned} S(v,n,\varepsilon)&=\sum_{j=1}^{k(n)}\Delta_j(v,n,\varepsilon)\\ &=E\left(\exp\left[i\sum_{m=1}^{k(n)}v_{n,m}Z_{n,m}\right]\right)-E\left(\exp\left[i\sum_{m=1}^{k(n)}v_{n,m}Y_{n,m}(\varepsilon)\right]\right). \end{aligned}$$

For $1\leqq k<s\leqq[k(n)-1]/2$, define

$$T(v,n,k,s,\varepsilon)=\sum_{j=s+1}^{k(n)-s}\Delta_j(v,n,\varepsilon).$$

Observe that, at this point, k appears only as a variable which determines the range of s. Now

$$|T(v,n,k,s,\varepsilon)-S(v,n,\varepsilon)|\leqq 2s\max_{1\leqq j\leqq k(n)}|\Delta_j(v,n,\varepsilon)|$$

and

$$\begin{aligned}|\Delta_j(v,n,\varepsilon)| &\leqq E(|\exp[iv_{n,j}Z_{n,j}]-1|)+E(|\exp[iv_{n,j}Y_{n,j}(\varepsilon)]-1|)\\ &\leqq V(E(|\hat{Z}_{n,j}(\varepsilon)|)+[E(|\hat{Z}_{n,j}(\varepsilon)|^2)]^{\frac{1}{2}})+4P(\{|Z_{n,j}|>\varepsilon\}),\end{aligned}$$

for $\|v\|\leqq V$ and hence, for every $\varepsilon>0$, $V>0$,

$$\max_{1\leqq j\leqq k(n)}\sup_{\|v\|\leqq V}|\Delta_j(v,n,\varepsilon)|\to 0 \qquad \text{as } n\to\infty$$

by Lemma 3 and *uan*. Consequently,

$$\lim_{n\to\infty}\sup_{\|v\|\leqq V}|T(v,n,k,s,\varepsilon)-S(v,n,\varepsilon)|=0$$

for every $s>k\geqq 1$, $\varepsilon>0$, $V>0$. Hence, it suffices to prove that

$$\lim_{\varepsilon\to 0}\limsup_{k\to\infty}\limsup_{s\to\infty}\limsup_{n\to\infty}\sup_{\|v\|\leqq V}|T(v,n,k,s,\varepsilon)|=0 \tag{5.2}$$

for every $V>0$.

Now suppose we can find a positive integer R and, for $r=1,2,\cdots,R$, a collection $\{\Delta_{r,j}(v,n,k,\varepsilon)\}$, such that, letting $\Delta_{0,j}(v,n,k,\varepsilon)=\Delta_j(v,n,\varepsilon)$,

$$\lim_{\varepsilon\to 0}\limsup_{k\to\infty}\limsup_{s\to\infty}\limsup_{n\to\infty}\sup_{\|v\|\leqq V}\left|\sum_{j=s+1}^{k(n)-s}[\Delta_{r,j}(v,n,k,\varepsilon)-\Delta_{r-1,j}(v,n,k,\varepsilon)]\right|=0 \tag{5.3}$$

for all $V>0$ and $r=1,2,\cdots,R$. Then it suffices to prove that, for all $V>0$,

$$\lim_{\varepsilon\to 0}\limsup_{k\to\infty}\limsup_{s\to\infty}\limsup_{n\to\infty}\sup_{\|v\|\leqq V}\left|\sum_{j=s+1}^{k(n)-s}\Delta_{R,j}(v,n,k,\varepsilon)\right|=0. \tag{5.4}$$

This step by step approximation can be accomplished in a variety of ways. By making small steps, R becomes larger but each approximation step is easier to verify. We let $R=8$ and define

$$\delta_{r,j}=\Delta_{r,j}(v,n,k,\varepsilon)-\Delta_{r-1,j}(v,n,k,\varepsilon) \qquad \text{for } r=1,2,\cdots,8,$$

in the manner listed below. To simplify the notation we write

$$A_j=E\left(\exp\left[i\sum_{m=1}^{j-1}v_{n,m}\tilde{Y}_{n,m}(\varepsilon)\right]\right)=\prod_{m=1}^{j-1}E(\exp[iv_{n,m}\tilde{Z}_{n,m}(\varepsilon)]),$$

$$B_j=E(\exp[iv_{n,j}\tilde{Z}_{n,j}(\varepsilon)]),\qquad C_j=E\left(\exp\left[i\sum_{m=j+1}^{k(n)}v_{n,m}Z_{n,m}\right]\right),$$

$$D_j = E\left(\exp\left[i\sum_{m=1}^{j-1} v_{n,m}\hat{Y}_{n,m}(\varepsilon)\right]\right) \quad \text{and} \quad E_{j,m} = v_{n,j}v_{n,m}E(\hat{Z}_{n,j}(\varepsilon)\hat{Z}_{n,m}(\varepsilon)).$$

Thus we have $\Delta_j(v, n, \varepsilon) = A_j(D_jC_{j-1} - D_{j+1}B_jC_j)$ and can define

$$\begin{aligned}
\delta_{1,j} = A_jD_jE\Bigg\{&(1-[v_{n,j}\hat{Z}_{n,j}(\varepsilon)]^2/2)\exp\left[iv_{n,j}\tilde{Z}_{n,j}(\varepsilon) + i\sum_{m=j+1}^{k(n)} v_{n,m}Z_{n,m}\right] \\
&+ iv_{n,j}\hat{Z}_{n,j}(\varepsilon)\left(1 + i\sum_{m=j+1}^{j+k} v_{n,m}\hat{Z}_{n,m}(\varepsilon)\right) \\
&\times \exp\left[i\sum_{m=j}^{j+k} v_{n,m}\tilde{Z}_{n,m}(\varepsilon) + i\sum_{m=j+k+1}^{k(n)} v_{n,m}Z_{n,m}\right]\Bigg\} \\
&- A_jD_jC_{j-1},
\end{aligned}$$

$$\begin{aligned}
\delta_{2,j} = A_jD_jE\Bigg\{&iv_{n,j}\hat{Z}_{n,j}(\varepsilon)\left(1 + i\sum_{m=j+1}^{j+k} v_{n,m}\hat{Z}_{n,m}(\varepsilon)\right)\exp\left(i\sum_{m=j+k+1}^{k(n)} v_{n,m}Z_{n,m}\right) \\
&\times\left(1 - \exp\left[i\sum_{m=j+1}^{j+k} v_{n,m}\tilde{Z}_{n,m}(\varepsilon)\right]\right)\Bigg\},
\end{aligned}$$

$$\begin{aligned}
\delta_{3,j} = A_jD_jE\Bigg\{&(B_j - \exp[iv_{n,j}\tilde{Z}_{n,j}(\varepsilon)])\Bigg[(1-[v_{n,j}\hat{Z}_{n,j}(\varepsilon)]^2/2)\exp\left(i\sum_{m=j+1}^{k(n)} v_{n,m}Z_{n,m}\right) \\
&+ iv_{n,j}\hat{Z}_{n,j}(\varepsilon)\left(1 + i\sum_{m=j+1}^{j+k} v_{n,m}\hat{Z}_{n,m}(\varepsilon)\right)\exp\left(i\sum_{m=j+k+1}^{k(n)} v_{n,m}Z_{n,m}\right)\Bigg]\Bigg\},
\end{aligned}$$

$$\delta_{4,j} = A_jB_jD_jE\left\{([v_{n,j}\hat{Z}_{n,j}(\varepsilon)]^2/2)\left[\exp\left(i\sum_{m=j+1}^{k(n)} v_{n,m}Z_{n,m}\right) - \exp\left(i\sum_{m=j+k+1}^{k(n)} v_{n,m}Z_{n,m}\right)\right]\right\},$$

$$\begin{aligned}
\delta_{5,j} = A_jB_jD_jE\Bigg\{&\left[C_{j+k} - \exp\left(i\sum_{m=j+k+1}^{k(n)} v_{n,m}Z_{n,m}\right)\right] \\
&\times iv_{n,j}\hat{Z}_{n,j}(\varepsilon)\left[1 + iv_{n,j}\hat{Z}_{n,j}(\varepsilon)/2 + i\sum_{m=j+1}^{j+k} v_{n,m}\hat{Z}_{n,m}(\varepsilon)\right]\Bigg\},
\end{aligned}$$

$$\delta_{6,j} = A_jB_jD_j(C_j - C_{j+k})E\left\{iv_{n,j}\hat{Z}_{n,j}(\varepsilon)\left[1 + iv_{n,j}\hat{Z}_{n,j}(\varepsilon)/2 + i\sum_{m=j+1}^{j+k} v_{n,m}\hat{Z}_{n,m}(\varepsilon)\right]\right\},$$

$$\begin{aligned}
\delta_{7,j} = A_jB_jC_j\Bigg\{D_{j+1} - D_j\Bigg[&1 + iv_{n,j}E(\hat{Z}_{n,j}(\varepsilon)) - E([v_{n,j}\hat{Z}_{n,j}(\varepsilon)]^2)/2 \\
&- \sum_{m=j-k}^{j-1} v_{n,j}v_{n,m}\operatorname{cov}(\hat{Z}_{n,j}(\varepsilon), \hat{Z}_{n,m}(\varepsilon))\Bigg]\Bigg\},
\end{aligned}$$

$$\delta_{8,j} = A_jB_jC_jD_j\sum_{m=j-k}^{j-1} v_{n,j}v_{n,m}E(\hat{Z}_{n,j}(\varepsilon))E(\hat{Z}_{n,m}(\varepsilon)).$$

Adding these steps and using $\hat{Z}_{n,j}(\varepsilon)\{\exp[iv_{n,j}\bar{Z}_{n,j}(\varepsilon)]-1\}\equiv 0$ gives us

$$\Delta_{8,j}(v,n,k,\varepsilon)=\Delta(v,n,\varepsilon)+\sum_{r=1}^{8}\delta_{r,j}=A_jB_jC_jD_j\sum_{m=1}^{k}(E_{j-m,j}-E_{j,j+m}).$$

Note that Bergström's proposed Gaussian components, as described in Section 4, were designed precisely to make $\Delta_{8,j}$ vanish identically and this appears to be necessary when the estimate

$$\left|\sum_{j=s+1}^{k(n)-s}\Delta_{8,j}\right|\leqq\sum_{j=s+1}^{k(n)-s}|\Delta_{8,j}|\leqq\sum_{j=s+1}^{k(n)-s}\left|\sum_{m=1}^{k}(E_{j-m,j}-E_{j,j+m})\right|$$

is used. Nevertheless, as we shall see, a summation by parts will allow us to establish the required limit:

$$\begin{aligned}\sum_{j=s+1}^{k(n)-s}\Delta_{8,j}&=\sum_{m=1}^{k}\sum_{j=s+1-m}^{k(n)-s-m}A_{j+m}B_{j+m}C_{j+m}D_{j+m}E_{j,j+m}-\sum_{m=1}^{k}\sum_{j=s+1}^{k(n)-s}A_jB_jC_jD_jE_{j,j+m}\\&=\sum_{m=1}^{k}\sum_{j=s+1-m}^{k(n)-s-m}[A_{j+m}B_{j+m}C_{j+m}D_{j+m}-A_jB_jC_jD_j]E_{j,j+m}\\&\quad+\sum_{m=1}^{k}\sum_{j=s+1-m}^{s}A_jB_jC_jD_jE_{j,j+m}-\sum_{m=1}^{k}\sum_{j=k(n)-s-m+1}^{k(n)-s}A_jB_jC_jD_jE_{j,j+m}.\end{aligned}$$

Now $|E_{j,j+m}|\leqq V^2\|\hat{Z}_{n,j}(\varepsilon)\|_2\|\hat{Z}_{n,j+m}(\varepsilon)\|_2$ for $\|v\|\leqq V$ and since A_j, B_j, C_j, and D_j are all characteristic functions they are each bounded in absolute value by one. Consequently

$$\sup_{\|v\|\leqq V}\left|\sum_{m=1}^{k}\sum_{j=s+1-m}^{s}A_jB_jC_jD_jE_{j,j+m}\right|\leqq k^2V^2\max_{1\leqq j\leqq k(n)}E([\hat{Z}_{n,j}(\varepsilon)]^2)\to 0 \tag{5.5}$$

as $n\to\infty$ for every $s>k\geqq 1$, $\varepsilon>0$, $V>0$, by Lemma 3. An identical estimate holds for

$$\sup_{\|v\|\leqq V}\left|\sum_{m=1}^{k}\sum_{j=k(n)-s-m+1}^{k(n)-s}A_jB_jC_jD_jE_{j,j+m}\right|.$$

Furthermore,

$$\sum_{j=s+1-m}^{k(n)-s-m}|E_{j,j+m}|\leqq V^2\sum_{j=1}^{k(n)}E([\hat{Z}_{n,j}(\varepsilon)]^2)$$

for $\|v\|\leqq V$, using the Cauchy–Schwarz inequality, and this sum is bounded as $n\to\infty$, for $s>k\geqq 1$, $\varepsilon>0$, $V>0$, by *ssb*. Hence it suffices to prove that, for every $s>k\geqq m\geqq 1$, $\varepsilon>0$, $V>0$,

$$\max_{1\leqq j\leqq k(n)-m}\sup_{\|v\|\leqq V}|A_{j+m}B_{j+m}C_{j+m}D_{j+m}-A_jB_jC_jD_j|\to 0 \tag{5.6}$$

as $n \to \infty$. But

$$\begin{aligned} &| A_{j+m}B_{j+m}C_{j+m}D_{j+m} - A_jB_jC_jD_j | \\ &\quad \leqq | A_{j+m} - A_j | + | B_{j+m} - B_j | + | C_{j+m} - C_j | + | D_{j+m} - D_j |, \end{aligned}$$

so it is enough to consider each term separately.

$$\begin{aligned} \sup_{\|v\| \leqq V} | A_{j+m} - A_j | &\leqq \sup_{\|v\| \leqq V} E\left(\sum_{r=j}^{j+m-1} | \exp[iv_{n,r}\tilde{y}_{n,r}(\varepsilon)] - 1 | \right) \\ &\leqq 2m \max_{1 \leqq j \leqq k(n)} P(\{| Z_{n,j} | > \varepsilon\}) \to 0 \end{aligned}$$

as $n \to \infty$ by *uan*. Similarly,

$$\begin{aligned} \sup_{\|v\| \leqq V} | B_{j+m} - B_j | &\leqq \sup_{\|v\| \leqq V} \{E(| \exp[iv_{n,j+m}\tilde{Y}_{n,j+m}(\varepsilon)] - 1 |) \\ &\qquad + E(| \exp[iv_{n,j}\tilde{Y}_{n,j}(\varepsilon)] - 1 |)\} \\ &\leqq 4 \max_{1 \leqq j \leqq k(n)} P(\{| Z_{n,j} | > \varepsilon\}). \end{aligned}$$

Also,

$$\begin{aligned} \sup_{\|v\| \leqq V} | C_{j+m} - C_j | &\leqq \sup_{\|v\| \leqq V} E\left(\sum_{r=j+1}^{j+m} | \exp(iv_{n,r}Z_{n,r}) - 1 | \right) \\ &\leqq m\left[V \max_{1 \leqq j \leqq k(n)} E(| \hat{Z}_{n,j}(\varepsilon) |) + 2 \max_{1 \leqq j \leqq k(n)} P(\{| Z_{n,j} | > \varepsilon\}) \right] \\ &\to 0 \end{aligned}$$

as $n \to \infty$, according to Lemma 3 and *uan*. Finally,

$$\begin{aligned} \sup_{\|v\| \leqq V} | D_{j+m} - D_j | &\leqq \sup_{\|v\| \leqq V} E\left(\sum_{r=j}^{j+m-1} | \exp[iv_{n,r}\hat{Y}_{n,r}(\varepsilon)] - 1 | \right) \\ &\leqq mV \max_{1 \leqq j \leqq k(n)} E(| \hat{Y}_{n,j}(\varepsilon) |) \leqq mV \max_{1 \leqq j \leqq k(n)} \| \hat{Z}_{n,j}(\varepsilon) \|_2 \\ &\to 0 \end{aligned}$$

as $n \to \infty$, again using Lemma 3. Therefore, we have the first result we are seeking, namely

$$\sup_{\|v\| \leqq V} \left| \sum_{j=s+1}^{k(n)-s} \Delta_{8,j}(v, n, k, \varepsilon) \right| \to 0 \qquad \text{as } n \to \infty \tag{5.7}$$

for every $s > k \geqq 1$, $\varepsilon > 0$, $V > 0$.

It remains to verify the legitimacy of each of our approximation steps $\delta_{r,j}$, $r=1,2,\cdots,8$. We begin with the expansions, valid for real θ,

$$\tag{5.8} \begin{gathered} |e^{i\theta}-1-i\theta+\theta^2/2|\leqq|\theta|^3/6,\\ |e^{i\theta}-1-i\theta|\leqq|\theta|^2/2. \end{gathered}$$

We let θ be $v_{n,j}\hat{Z}_{n,j}(\varepsilon)$ in the first of these and then equal $\sum_{m=j+1}^{j+k} v_{n,m}\hat{Z}_{n,m}(\varepsilon)$ in the second to obtain the estimate, valid for every $V>0$,

$$\begin{aligned} \sup_{\|v\|\leqq V}\left|\sum_{j=s+1}^{k(n)-s}\delta_{1,j}\right| &\leqq \sum_{j=s+1}^{k(n)-s} V^3\left\{E(|\hat{Z}_{n,j}(\varepsilon)|^3)/6\right.\\ &\qquad \left.+E\left(|\hat{Z}_{n,j}(\varepsilon)|\left[\sum_{m=j+1}^{j+k}|\hat{Z}_{n,m}(\varepsilon)|\right]^2\right)\Big/2\right\}\\ &\leqq (V^3/6)\sum_{j=1}^{k(n)}E(|\hat{Z}_{n,j}(\varepsilon)|^3)\\ &\quad+(V^3/2)\sum_{m=1}^{k}\sum_{j=1}^{k(n)-m}E(|\hat{Z}_{n,j}(\varepsilon)|\,|\hat{Z}_{j+m}(\varepsilon)|^2)\\ &\quad+V^3\sum_{r=1}^{k-1}\sum_{m=1}^{k-r}\sum_{j=1}^{k(n)-m-r}E(|\hat{Z}_{n,j}(\varepsilon)\hat{Z}_{n,j+m}\hat{Z}_{n,j+m+r}(\varepsilon)|). \end{aligned}$$

By (4.2) the first sum approaches zero as $n\to\infty$ and then $\varepsilon\to 0$ and does not depend upon either s or k. The second and third sums do not depend upon s and, for each $k\geqq 1$ and $\varepsilon>0$, they approach zero as $n\to\infty$, by Lemma 12.

$$\begin{aligned} \sup_{\|v\|\leqq V}\left|\sum_{j=s+1}^{k(n)-s}\delta_{2,j}\right| &\leqq V(1+kV\varepsilon)\sum_{j=s+1}^{k(n)-s}E\left(|\hat{Z}_{n,j}(\varepsilon)|\sum_{m=j+1}^{j+k}|\exp[iv_{n,m}\tilde{Z}_{n,m}(\varepsilon)]-1|\right)\\ &\leqq 2V(1+kV\varepsilon)\sum_{m=1}^{k}\sum_{j=1}^{k(n)-m}\int_{\{|Z_{n,j+m}|>\varepsilon\}}|\hat{Z}_{n,j}(\varepsilon)|\,dP\to 0 \quad \text{as } n\to\infty \end{aligned}$$

for every $s>k\geqq 1$, $\varepsilon>0$, $V>0$ using Lemma 11.

$$\begin{aligned} \sup_{\|v\|\leqq V}\left|\sum_{j=s+1}^{k(n)-s}\delta_{3,j}\right| &\leqq \sum_{j=s+1}^{k(n)-s}\sup_{\|v\|\leqq V}\left|E\left\{(B_j-\exp[iv_{n,j}\tilde{Z}_{n,j}(\varepsilon)])\exp\left(i\sum_{m=j+k+1}^{k(n)}v_{n,m}Z_{n,m}\right)\right\}\right|\\ &+\sum_{j=s+1}^{k(n)-s}\sup_{\|v\|\leqq V}\left|E\left\{(B_j-\exp[iv_{n,j}\tilde{Z}_{n,j}(\varepsilon)])\left(\exp\left[i\sum_{m=j+1}^{k(n)}v_{n,m}Z_{n,m}\right]\right.\right.\right.\\ &\qquad\left.\left.\left.-\exp\left[i\sum_{m=j+k+1}^{k(n)}v_{n,m}Z_{n,m}\right]\right)\right\}\right|\\ &+V[1+(1+2k)V\varepsilon/2]\sum_{j=s+1}^{k(n)-s}E(|\exp[iv_{n,j}\tilde{Z}_{n,j}(\varepsilon)]-1|)E(|\hat{Z}_{n,j}(\varepsilon)|). \end{aligned}$$

We deal with each of these sums separately. By means of (4.3), for $p = 1$, $q = \infty$, and complex valued random variables, the first sum is dominated by

$$\sum_{j=s+1}^{k(n)-s} \sup_{\|v\|\leqq V} 8\phi(n, k+1) E(|\exp[iv_{n,j}\tilde{Z}_{n,j}(\varepsilon)] - 1|)$$

$$\leqq 16\phi(n, k+1) \sum_{j=1}^{k(n)} P(\{|Z_{n,j}| > \varepsilon\}).$$

This sum does not depend upon s nor V and, for every $\varepsilon > 0$, it approaches zero as $n \to \infty$ and then $k \to \infty$, by *stb* and the definition of ϕ-mixing.

The second sum is dominated by

$$\sum_{j=s+1}^{k(n)-s} \sup_{\|v\|\leqq V} |B_j - 1| E\Big(\sum_{m=j+1}^{j+k} |\exp(iv_{n,m}Z_{n,m}) - 1|\Big)$$

$$+ \sum_{j=s+1}^{k(n)-s} \sup_{\|v\|\leqq V} E\Big(|\exp[iv_{n,j}\tilde{Z}_{n,j}(\varepsilon)] - 1| \sum_{m=j+1}^{j+k} |\exp(iv_{n,m}Z_{n,m}) - 1|\Big)$$

$$\leqq \sum_{m=1}^{k} \sum_{j=1}^{k(n)-m} 2P(\{|Z_{n,j}| > \varepsilon\})[VE(|\hat{Z}_{n,j+m}(\varepsilon)|) + 2P(\{|Z_{n,j+m}| > \varepsilon\})]$$

$$+ \sum_{m=1}^{k} \sum_{j=1}^{k(n)-m} \Big[2V \int_{\{|Z_{n,j}|>\varepsilon\}} |\hat{Z}_{n,j+m}(\varepsilon)| \, dP$$

$$+ 4P(\{|Z_{n,j}| > \varepsilon, |Z_{n,j+m}| > \varepsilon\})\Big]$$

which approaches zero as $n \to \infty$ for every $s > k \geqq 1$, $\varepsilon > 0$, $V > 0$, because of *uan*, *stb*, *sjn*, Lemma 3, and Lemma 11. The third sum is the simplest, since it is bounded above by

$$2V[1 + (1+2k)V\varepsilon/2] \sum_{j=1}^{k(n)} P(\{|Z_{n,j}| > \varepsilon\}) E(|\hat{Z}_{n,j}(\varepsilon)|)$$

and this approaches zero as $n \to \infty$ for every $s > k \geqq 1$, $\varepsilon > 0$, $V > 0$ by *stb* and Lemma 3.

$$\sup_{\|v\|\leqq V} \Big|\sum_{j=s+1}^{k(n)-s} \delta_{4,j}\Big| \leqq (V^2/2) \sum_{m=1}^{k} \sum_{j=1}^{k(n)-m} \Big[VE(|\hat{Z}_{n,j}(\varepsilon)|^2 |\hat{Z}_{n,j+m}(\varepsilon)|)$$

$$+ 2\int_{\{|Z_{n,j+m}|>\varepsilon\}} |\hat{Z}_{n,j}(\varepsilon)|^2 \, dP\Big]$$

which approaches zero as $n \to \infty$ for every $s > k \geqq 1$, $\varepsilon > 0$, $V > 0$ by Lemma 11 and Lemma 12.

$$\sup_{\|v\|\leqq V}\left|\sum_{j=s+1}^{k(n)-s}\delta_{5,j}\right|$$

$$\leqq \sum_{j=s+1}^{k(n)-s}\sup_{\|v\|\leqq V}\left|E\left(\left[C_{j+k}-\exp\left(i\sum_{m=j+k+1}^{k(n)}v_{n,m}Z_{n,m}\right)\right]v_{n,j}\hat{Z}_{n,j}(\varepsilon)\right)\right|$$

$$+\sum_{j=s+1}^{k(n)-s}\sup_{\|v\|\leqq V}\left|E\left(\left[C_{j+k}-\exp\left(i\sum_{m=j+k+1}^{k(n)}v_{n,m}Z_{n,m}\right)\right][v_{n,j}\hat{Z}_{n,j}(\varepsilon)]^2/2\right)\right|$$

$$+\sum_{j=s+1}^{k(n)-s}\sup_{\|v\|\leqq V}\left|E\left(\left[C_{j+k}-\exp\left(i\sum_{m=j+k+1}^{k(n)}v_{n,m}Z_{n,m}\right)\right]\sum_{m=j+1}^{j+k}v_{n,j}v_{n,m}\hat{Z}_{n,j}(\varepsilon)\hat{Z}_{n,m}(\varepsilon)\right)\right|$$

Again we shall deal with these sums one at a time. The first is dominated by

$$\sum_{j=s+1}^{k(n)-s}\sup_{\|v\|\leqq V}\left|\sum_{m=k+1}^{k(n)-j}E\left(v_{n,j}\hat{Z}_{n,j}(\varepsilon)\left[C_{j+m-1}-C_{j+m}\right.\right.\right.$$

$$\left.\left.\left.-\exp\left(i\sum_{r=j+m+1}^{k(n)}v_{n,r}Z_{n,r}\right)(\exp[iv_{n,j+m}Z_{n,j+m}]-1)\right]\right)\right|$$

$$\leqq \sum_{j=s+1}^{k(n)-s}\sup_{\|v\|\leqq V}\sum_{m=k+1}^{k(n)-j}4[\phi(n,m)]^{\frac12}\|v_{n,j}\hat{Z}_{n,j}(\varepsilon)\|_2\|\exp[iv_{n,j+m}Z_{n,j+m}]-1\|_2$$

$$\leqq \sum_{j=k+1}^{k(n)-k-1}4V\|\hat{Z}_{n,j}(\varepsilon)\|_2\sum_{m=k+1}^{k(n)-j}[\phi(n,m)]^{\frac12}\{V\|\hat{Z}_{n,j+m}(\varepsilon)\|_2+2[P(\{|Z_{n,j+m}|>\varepsilon\})]^{\frac12}\}$$

$$\leqq 4V\sum_{m=k+1}^{k(n)-1}[\phi(n,m)]^{\frac12}\left\{V\sum_{j=1}^{k(n)}E([\hat{Z}_{n,j}(\varepsilon)]^2)+2\left[\sum_{j=1}^{k(n)}E([\hat{Z}_{n,j}(\varepsilon)]^2)\right]^{\frac12}\right.$$

$$\left.\times\left[\sum_{j=1}^{k(n)}P(\{|Z_{n,j}|>\varepsilon\})\right]^{\frac12}\right\}$$

for every $\varepsilon>0$, $V>0$, using (4.3) for $p=q=2$ and one factor complex valued. But the last sum does not depend upon s and approaches zero as $n\to\infty$ and then $k\to\infty$, according to *stb*, *ssb*, and Ibragimov's condition.

The second sum can be estimated by

$$2V^2\phi(n,k+1)\sum_{j=1}^{k(n)}E([\hat{Z}_{n,j}(\varepsilon)]^2)$$

for every $\varepsilon>0$, $V>0$, this time using (4.3) for $p=1$, $q=\infty$ with one complex factor. This expression does not depend upon s and tends to zero as $n\to\infty$ and then $k\to\infty$, using *ssb* and the definition of ϕ-mixing.

For the third sum we get the bound

$$\sum_{j=s+1}^{k(n)-s} \sup_{\|v\|\leqq V} \left| \sum_{m=1}^{k} v_{n,j}v_{n,j+m}E(\hat{Z}_{n,j}(\varepsilon)\hat{Z}_{n,j+m}(\varepsilon)\Big[C_{j+k} - C_{j+s-1} + C_{j+s-1} - \exp\Big(i\sum_{r=j+s}^{k(n)} v_{n,r}Z_{n,r}\Big)\Big(\exp\Big[i\sum_{r=j+k+1}^{j+s-1} v_{n,r}Z_{n,r}\Big] - 1 + 1\Big)\Big]\Big) \right|$$

$$\leqq V^2 \sum_{j=s+1}^{k(n)-s} \sum_{m=1}^{k} \sum_{r=k+1}^{s-1} \sup_{\|v\|\leqq V} E(|\exp(iv_{n,j+r}Z_{n,j+r}) - 1|)E(|\hat{Z}_{n,j}(\varepsilon)\hat{Z}_{n,j+m}(\varepsilon)|)$$

$$+ V^2 \sum_{j=s+1}^{k(n)-s} \sum_{m=1}^{k} \sum_{r=k+1}^{s-1} \sup_{\|v\|\leqq V} E(|\exp(iv_{n,j+r}Z_{n,j+r}) - 1|\,|\hat{Z}_{n,j}(\varepsilon)\hat{Z}_{n,j+m}(\varepsilon)|)$$

$$+ V^2 \sum_{j=s+1}^{k(n)-s} \sum_{m=1}^{k} 4\phi(n, s-m)E(|\hat{Z}_{n,j}(\varepsilon)\hat{Z}_{n,j+m}(\varepsilon)|)$$

$$\leqq skV^2 \sum_{j=1}^{k(n)} E([\hat{Z}_{n,j}(\varepsilon)]^2)\Big[V \max_{1\leqq j\leqq k(n)} E(|\hat{Z}_{n,j}(\varepsilon)|) + 2 \max_{1\leqq j\leqq k(n)} P(\{|Z_{n,j}| > \varepsilon\})\Big]$$

$$+ V^3 \sum_{m=1}^{k} \sum_{r=k+1}^{s-1} \sum_{j=1}^{k(n)-r} E(|\hat{Z}_{n,j}(\varepsilon)\hat{Z}_{n,j+m}(\varepsilon)\hat{Z}_{n,j+r}(\varepsilon)|)$$

$$+ 2V^2k\varepsilon \sum_{r=k+1}^{s-1} \sum_{j=1}^{k(n)-r} \int_{\{|Z_{n,j+r}|>\varepsilon\}} |\hat{Z}_{n,j}(\varepsilon)|\,dP$$

$$+ 4kV^2 \sum_{j=1}^{k(n)} E([\hat{Z}_{n,j}(\varepsilon)]^2)\Big[\max_{1\leqq m\leqq k} \phi(n, s-m)\Big].$$

Now with the exception of the last term, each of these sums approaches zero as $n \to \infty$ for every $s > k \geqq 1$, $\varepsilon > 0$, $V > 0$, via *uan*, *ssb*, Lemma 3, Lemma 11, and Lemma 12. The last sum, obtained from (4.3) with $p = 1$, $q = \infty$ and one complex factor, approaches zero for every $k \geqq 1$, $\varepsilon > 0$, $V > 0$ as $n \to \infty$ and then $s \to \infty$, again using *ssb* and the definition of ϕ-mixing:

$$\sup_{\|v\|\leqq V} \left| \sum_{j=s+1}^{k(n)-s} \delta_{6,j} \right| \leqq \sum_{j=s+1}^{k(n)-s} \sup_{\|v\|\leqq V} \sum_{m=1}^{k} |C_{j+m-1} - C_{j+m}| \Big\{ V|E(\hat{Z}_{n,j}(\varepsilon))| + V^2E([\hat{Z}_{n,j}(\varepsilon)]^2)/2 + V^2 \sum_{m=1}^{k} |E(\hat{Z}_{n,j}(\varepsilon)\hat{Z}_{n,j+m}(\varepsilon))| \Big\}$$

$$\leqq k\Big[V \max_{1\leqq j\leqq k(n)} E(|\hat{Z}_{n,j}(\varepsilon)|) + 2 \max_{1\leqq j\leqq k(n)} P(\{|Z_{n,j}| > \varepsilon\})\Big] \times \Big\{ V\sum_{j=1}^{k(n)} |E(\hat{Z}_{n,j}(\varepsilon))| + V^2(1+2k)/2 \sum_{j=1}^{k(n)} E([\hat{Z}_{n,j}(\varepsilon)]^2) \Big\}$$

$$\to 0$$

as $n \to \infty$ for every $s > k \geqq 1$, $\varepsilon > 0$, $V > 0$ by Lemma 3, *uan*, *smb*, and *ssb*.

In the next step, we deal with the Gaussian components $\{\hat{Y}_{n,j}(\varepsilon)\}$ and so we need the relation, valid for (X, Y) with a 2-dimensional normal distribution,

$$(5.9)\qquad E(\exp[i(X+Y)]) = E(\exp(iX))E(\exp(iY))\exp[-\operatorname{cov}(X,Y)].$$

Consequently,

$$D_{j+1} = D_j E(\exp[iv_{n,j}\hat{Y}_{n,j}(\varepsilon)])\exp\left[-\sum_{m=1}^{j-1} v_{n,j}v_{n,m}\operatorname{cov}(\hat{Z}_{n,j}(\varepsilon),\hat{Z}_{n,m}(\varepsilon))\right]$$

and thus

$$\sup_{\|v\|\leqq V}\left|\sum_{j=s+1}^{k(n)-s}\delta_{7,j}\right|$$

$$\leqq \sum_{j=s+1}^{k(n)-s}\sup_{\|v\|\leqq V}\exp\left[-\sum_{m=j-k}^{j-1} v_{n,j}v_{n,m}\operatorname{cov}(\hat{Z}_{n,j}(\varepsilon),\hat{Z}_{n,m}(\varepsilon))\right]$$

$$\times\left|\exp\left[-\sum_{m=1}^{j-k-1} v_{n,j}v_{n,m}\operatorname{cov}(\hat{Z}_{n,j}(\varepsilon),\hat{Z}_{n,m}(\varepsilon))\right]-1\right|$$

$$+\sum_{j=s+1}^{k(n)-s}\sup_{\|v\|\leqq V}\left|E(\exp[iv_{n,j}\hat{Y}_{n,j}(\varepsilon)])\exp\left[-\sum_{m=j-k}^{j-1} v_{n,j}v_{n,m}\operatorname{cov}(\hat{Z}_{n,j}(\varepsilon),\hat{Z}_{n,m}(\varepsilon))\right]\right.$$

$$-\left[1+iv_{n,j}E(\hat{Z}_{n,j}(\varepsilon))-E([v_{n,j}\hat{Z}_{n,j}(\varepsilon)]^2)/2\right.$$

$$\left.\left.-\sum_{m=j-k}^{j-1} v_{n,j}v_{n,m}\operatorname{cov}(\hat{Z}_{n,j}(\varepsilon),\hat{Z}_{n,m}(\varepsilon))\right]\right|.$$

Next, observe that

$$\sup_{\|v\|\leqq V}\left|\sum_{m=j-k}^{j-1} v_{n,j}v_{n,m}\operatorname{cov}(\hat{Z}_{n,j}(\varepsilon),\hat{Z}_{n,m}(\varepsilon))\right|\leqq kV^2\max_{1\leqq j\leqq k(n)}E([\hat{Z}_{n,j}(\varepsilon)]^2)$$

and

$$\sup_{\|v\|\leqq V}\left|\sum_{m=1}^{j-k-1} v_{n,j}v_{n,m}\operatorname{cov}(\hat{Z}_{n,j}(\varepsilon),\hat{Z}_{n,m}(\varepsilon))\right|$$

$$\leqq 2V^2\max_{1\leqq j\leqq k(n)}E([\hat{Z}_{n,j}(\varepsilon)]^2)\sum_{m=k+1}^{k(n)-1}[\phi(n,m)]^{\frac{1}{2}},$$

with each of these bounds approaching zero as $n \to \infty$, for every $s > k \geqq 1$, $\varepsilon > 0$, $V > 0$, using Lemma 3 and Ibragimov's condition. Consequently, for n sufficiently large, the first sum in our estimate cannot exceed

$$\exp\left\{kV^2 \max_{1\leqq j\leqq k(n)} E([\hat{Z}_{n,j}(\varepsilon)]^2)\right\}$$

$$\times 4V^2 \sum_{j=k+2}^{k(n)} \sum_{m=k+1}^{j-1} [\phi(n,m)]^{\frac{1}{2}} \| \hat{Z}_{n,j}(\varepsilon)\|_2 \| \hat{Z}_{n,j-m}(\varepsilon)\|_2$$

$$\leqq \exp\left\{kV^2 \max_{1\leqq j\leqq k(n)} E([\hat{Z}_{n,j}(\varepsilon)]^2)\right\}$$

$$\times 4V^2 \sum_{m=k+1}^{k(n)-1} [\phi(n,m)]^{\frac{1}{2}} \sum_{j=1}^{k(n)} E([\hat{Z}_{n,j}(\varepsilon)]^2)$$

which does not depend upon s and approaches zero as $n\to\infty$ and then $k\to\infty$, for every $\varepsilon>0$, $V>0$, by the above remark, *ssb*, and Ibragimov's condition.

For the second sum we temporarily introduce the notation

$$S_1 = iv_{n,j}E(\hat{Z}_{n,j}(\varepsilon)) - E([v_{n,j}\hat{Z}_{n,j}(\varepsilon)]^2)/2,$$

$$S_2 = -\sum_{m=j-k}^{j-1} v_{n,j}v_{n,m}\operatorname{cov}(\hat{Z}_{n,j}(\varepsilon), \hat{Z}_{n,m}(\varepsilon)),$$

$$E_1 = E(\exp[iv_{n,j}\hat{Y}_{n,j}(\varepsilon)]), \qquad E_2 = \exp S_2,$$

$$R_1 = |E_1 - (1+S_1)| \quad\text{and}\quad R_2 = |E_2 - (1+S_2)|.$$

Then we need to estimate $\sum_{j=s+1}^{k(n)-s} \sup_{\|v\|\leqq V} |E_1E_2 - (1+S_1+S_2)|$, where

$$|E_1E_2 - (1+S_1+S_2)| \leqq R_2 + |1+S_2|R_1 + |S_1S_2|,$$

since $|E_1|\leqq 1$. Now since $S_2\to 0$ as $n\to\infty$, we see that for n sufficiently large and $\|v\|\leqq V$

$$R_2 \leqq |S_2|^2 \leqq k^2V^4E([\hat{Z}_{n,j}(\varepsilon)]^2) \max_{1\leqq j\leqq k(n)} E([\hat{Z}_{n,j}(\varepsilon)]^2).$$

Furthermore, for $\|v\|\leqq V$

$$\begin{aligned} R_1 &\leqq V^3E(|\hat{Y}_{n,j}(\varepsilon)|^3) \\ &\leqq 4V^3[E(|\hat{Y}_{n,j}(\varepsilon) - \nu_{n,j}(\varepsilon)|^3) + |E(\hat{Z}_{n,j}(\varepsilon))|^3] \\ &\leqq 4V^3\{4[E([\hat{Z}_{n,j}(\varepsilon)]^2)]^{\frac{3}{2}}/(2\pi)^{\frac{1}{2}} + |E(\hat{Z}_{n,j}(\varepsilon))|^3\}. \end{aligned}$$

Consequently, for $s>k\geqq 1$, $\varepsilon>0$, $V>0$ we see that as $n\to\infty$

$$\sum_{j=s+1}^{k(n)-s} \sup_{\|v\|\leqq V} |E_1E_2 - (1+S_1+S_2)| \to 0,$$

where we make use of Lemma 3, *smb*, and *ssb*.

$$\sup_{\|v\|\leqq V}\left|\sum_{j=s+1}^{k(n)-s}\delta_{8,j}\right|\leqq kV^2\max_{1\leqq j\leqq k(n)}E(|\hat{Z}_{n,j}(\varepsilon)|)\sum_{j=1}^{k(n)}|E(\hat{Z}_{n,j}(\varepsilon))|$$

which approaches zero as $n\to\infty$ for $s>k\geqq 1$, $\varepsilon>0$, $V>0$, according to Lemma 3 and *smb*. This completes the proof of Proposition A.

Corollary 1 is implied by the assertion that for every positive integer M, every partition $0=t_0<t_1<\cdots<t_M\leqq 1$ of the unit interval, and every $V>0$, we have

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\sup_{\|v\|\leqq V}\left|E\left(\exp\left\{i\sum_{m=1}^{M}v_m[U_n(t_m)-U_n(t_{m-1})]\right\}\right)\right.$$
$$\left.-E\left(\exp\left\{i\sum_{m=1}^{M}v_m[T_n(t_m,\varepsilon)-T_n(t_{m-1},\varepsilon)]\right\}\right)\right|=0,$$

where $v=(v_1,v_2,\cdots,v_M)$ and $\|v\|=\max_{1\leqq m\leqq M}|v_m|$. This follows immediately from Proposition A if we define $v_{n,j}=v_m$ for $[k(n)\theta_n(t_{m-1})]<j\leqq[k(n)\theta_n(t_m)]$ with $v_{n,j}=0$ for other values of j. Corollary 2 is a special case of this, letting $M=1$ and $t_1=1$. In Corollary 3 we have just added the same centering term to each of the sums we are comparing.

6. Proofs of the other propositions and their corollaries

PROPOSITION B. $\{\tilde{W}_{n,j}(\varepsilon)\}$ satisfies *uan*, since for $u>0$

$$P(\{|\tilde{W}_{n,j}(\varepsilon)|>u\})=\begin{cases}P(\{|X_{n,j}|>u\}, & |u|>\varepsilon,\\ P(\{|X_{n,j}|>\varepsilon\}, & 0<|u|\leqq\varepsilon.\end{cases}\tag{6.1}$$

Similarly, we see that $\{\tilde{W}_{n,j}(\varepsilon)\}$ satisfies condition (i) of the main theorem for the function Q_ε. Furthermore, for $0<\varepsilon'\leqq\varepsilon$,

$$\tilde{W}_{n,j}(\varepsilon)I(\{|\tilde{W}_{n,j}(\varepsilon)|\leqq\varepsilon'\})=0$$

and hence

$$\lim_{\varepsilon'\to 0}\limsup_{n\to\infty}\sum_{j=1}^{k(n)}\mathrm{Var}[\tilde{W}_{n,j}(\varepsilon)I(\{|\tilde{W}_{n,j}(\varepsilon)|\leqq\varepsilon'\})]=0.\tag{6.2}$$

Consequently, by the classical theorem for sums of independent random variables,

$$\sum_{j=1}^{k(n)}\left[\tilde{W}_{n,j}(\varepsilon)-\int_{\{|\tilde{W}_{n,j}(\varepsilon)|\leqq\varepsilon_0\}}\tilde{W}_{n,j}(\varepsilon)dP\right]=\sum_{j=1}^{k(n)}[\tilde{W}_{n,j}(\varepsilon)-\mu_{n,j}(\varepsilon_0)+\mu_{n,j}(\varepsilon)]$$

converges in distribution as $n\to\infty$ to the infinitely divisible law determined in the form (2.5) by Q_ε, $\gamma_\varepsilon(\varepsilon_0)=0$, and $\sigma^2_\varepsilon=0$.

COROLLARY. By means of Lemma 3 and Lemma 8, we see that $\{X_{n,j} - \mu_{n,j}(\varepsilon_0)\}$ satisfies *uan* and condition (i), with the same Q as for $\{X_{n,j}\}$. Hence $\sum_{j=1}^{k(n)}[\tilde{Y}_{n,j}(\varepsilon) - \nu_{n,j}(\varepsilon_0) + \nu_{n,j}(\varepsilon)]$ converges in distribution as $n \to \infty$ to the infinitely divisible law determined in the form (2.5) by Q_ε, $\gamma_\varepsilon(\varepsilon_0) = 0$, and $\sigma_\varepsilon^2 = 0$. However, by Lemma 9,

$$\left|\sum_{j=1}^{k(n)} \nu_{n,j}(\varepsilon_0)\right| \leqq \sum_{j=1}^{k(n)} |\nu_{n,j}(\varepsilon_0)| \to 0 \qquad \text{as } n \to \infty.$$

PROPOSITION C.

$$\begin{aligned} E\Big(\exp\Big[iv\sum_{j=1}^{k(n)} Y_{n,j}(\varepsilon)\Big]\Big) &= E\Big(\exp\Big[iv\sum_{j=1}^{k(n)} \hat{Y}_{n,j}(\varepsilon)\Big]\Big)E\Big(\exp\Big[iv\sum_{j=1}^{k(n)} \tilde{Y}_{n,j}(\varepsilon)\Big]\Big) \\ &= \exp\Big\{-v^2\operatorname{var}\Big[\sum_{j=1}^{k(n)} \hat{Z}_{n,j}(\varepsilon)\Big]\Big/2\Big\}E\Big(\exp\Big\{iv\sum_{j=1}^{k(n)}[\tilde{Y}_{n,j}(\varepsilon) + \nu_{n,j}(\varepsilon)]\Big\}\Big) \end{aligned}$$

and we know that the second factor converges as $n \to \infty$, uniformly for $|v| \leqq V$, to $\exp[\psi_\varepsilon(v)]$, the characteristic function of the infinitely divisible law determined by Q_ε, $\gamma_\varepsilon(\varepsilon_0) = 0$, and $\sigma_\varepsilon^2 = 0$. Furthermore, if $\exp[\psi_0(v)]$ is the characteristic function of the infinitely divisible law determined in the form (2.5) by Q, $\gamma_0(\varepsilon_0) = 0$, and $\sigma_0^2 = 0$, then for $|v| \leqq V$

$$|\psi_\varepsilon(v) - \psi_0(v)| = \left|\int_{\{0<|u|\leqq\varepsilon\}} (e^{iuv} - 1 - iuv)dQ(u)\right| \leqq (V^2/2)\int_{\{0<|u|\leqq\varepsilon\}} u^2 dQ(u)$$

which approaches zero as $\varepsilon \to 0$. Since $\{Z_{n,j}\}$ satisfies condition (iii), we have

$$\begin{aligned} &\Big|\exp\Big\{-v^2\operatorname{var}\Big[\sum_{j=1}^{k(n)} \hat{Z}_{n,j}(\varepsilon)\Big]\Big/2\Big\}E\Big(\exp\Big\{iv\sum_{j=1}^{k(n)}[\tilde{Y}_{n,j}(\varepsilon) + \nu_{n,j}(\varepsilon)]\Big\}\Big) \\ &\qquad\qquad - \exp[-(v^2\sigma^2/2) + \psi_0(v)]\Big| \\ &\leqq \Big|E\Big(\exp\Big\{iv\sum_{j=1}^{k(n)}[\tilde{Y}_{n,j}(\varepsilon) + \nu_{n,j}(\varepsilon)]\Big\}\Big) - \exp[\psi_\varepsilon(v)\Big| \\ &\qquad + |\exp[\psi_\varepsilon(v)] - \exp[\psi_0(v)]| + (v^2/2)\Big|\operatorname{var}\Big[\sum_{j=1}^{k(n)} \hat{Z}_{n,j}(\varepsilon)\Big] - \sigma^2\Big| \end{aligned}$$

which approaches zero, uniformly for $|v| \leqq V$, as $n \to \infty$ and then $\varepsilon \to 0$ through positive values such that $\pm\varepsilon$ are continuity points of Q. The limit is the characteristic function of the infinitely divisible law determined in the form (2.5) by Q, $\gamma_0(\varepsilon_0) = 0$, and σ^2.

COROLLARY.

$$\left|E\left(\exp\left\{iv\left[\sum_{j=1}^{k(n)} Y_{n,j}(\varepsilon)+\sum_{j=1}^{k(n)} \mu_{n,j}(\varepsilon_0)-A_n\right]\right\}\right)-\exp\left[iv\gamma(\varepsilon_0)-(v^2\sigma^2/2)+\psi_0(v)\right]\right|$$

$$\leqq |v|\left|\sum_{j=1}^{k(n)} \mu_{n,j}(\varepsilon_0)-A_n-\gamma(\varepsilon_0)\right|$$

$$+\left|E\left(\exp\left[iv\sum_{j=1}^{k(n)} Y_{n,j}(\varepsilon)\right]\right)-\exp\left[-(v^2\sigma^2/2)+\psi_0(v)\right]\right|$$

which approaches zero, uniformly for $|v|\leqq V$, as $n\to\infty$ and then $\varepsilon\to 0$ through positive values such that $\pm\varepsilon$ are continuity points of Q. In this case, the limit is the characteristic function of the infinitely divisible law determined in the form (2.5) by Q, $\gamma(\varepsilon_0)$, and σ^2.

PROPOSITION D. If $\hat{F}$ and $\hat{G}$ are characteristic functions, let

$$d(\hat{F},\hat{G})=\sum_{m=1}^{\infty} 2^{-m}\sup_{|v|\leqq m}|\hat{F}(v)-\hat{G}(v)|, \tag{6.3}$$

so that convergence in the metric d is equivalent to uniform convergence on compact subsets of the real line and hence to convergence in distribution. Thus if $\varepsilon_0>0$ is arbitrary, $\hat{F}_{n,\varepsilon}$ is the characteristic function of $\Sigma_{j=1}^{k(n)} Y_{n,j}(\varepsilon)+\Sigma_{j=1}^{k(n)}\mu_{n,j}(\varepsilon_0)-A_n$, and $\hat{G}$ is the characteristic function of the limit distribution of $\Sigma_{j=1}^{k(n)} X_{n,j}-A_n$, we know from Corollary 3 to Proposition A that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} d(\hat{F}_{n,\varepsilon},\hat{G})=0. \tag{6.4}$$

Let $S(n,\varepsilon)=\Sigma_{j=1}^{k(n)} E([\hat{Z}_{n,j}(\varepsilon)]^2)$ and $V(n,\varepsilon)=\mathrm{var}[\Sigma_{j=1}^{k(n)}\hat{Z}_{n,j}(\varepsilon)]$. Then by Lemma 6 and Lemma 16 we see that $S(n,\varepsilon)$ and $V(n,\varepsilon)$ are bounded for $n\geqq 1$ and $0<\varepsilon\leqq\varepsilon_0$. Let $\{n(m)\}$ be a strictly increasing sequence of positive integers and define

$$S'(\varepsilon)=\limsup_{m\to\infty} S(n(m),\varepsilon),\qquad S'=\limsup_{\varepsilon\to 0} S'(\varepsilon). \tag{6.5}$$

Let $\{\varepsilon_r\}$ be a sequence, decreasing to zero as $r\to\infty$, such that

$$\lim_{r\to\infty} S'(\varepsilon_r)=S'. \tag{6.6}$$

Now choose $\{m(r)\}$ to be a strictly increasing sequence of positive integers such that

$$\lim_{r\to\infty} S(n[m(r)],\varepsilon_r)=S' \quad\text{and}\quad \lim_{r\to\infty} d(\hat{F}_{n[m(r)],\varepsilon_r},\hat{G})=0. \tag{6.7}$$

By passing to a subsequence of $(n[m(r)], \varepsilon_r)$ if necessary, we can assume that the bounded sequence $\{V(n[m(r)], \varepsilon_r)\}$ converges, say

$$\lim_{r\to\infty} V(n[m(r)], \varepsilon_r) = \sigma^2 \geqq 0. \tag{6.8}$$

Since

$$\hat{F}_{n,\varepsilon}(v) = \exp[-v^2 V(n,\varepsilon)/2] \times E\left(\exp\left[iv\left\{\sum_{j=1}^{k(n)} [\tilde{Y}_{n,j}(\varepsilon) + v_{n,j}(\varepsilon)] + \sum_{j=1}^{k(n)} \mu_{n,j}(\varepsilon_0) - A_n\right\}\right]\right),$$

we see that, for every v, the characteristic function of

$$\sum_{j=1}^{k(n[m(r)])} [\tilde{Y}_{n[m(r)],j}(\varepsilon_r) + \nu_{n[m(r)],j}(\varepsilon_r)] + \sum_{j=1}^{k(n[m(r)])} \mu_{n[m(r)],j}(\varepsilon_0) - A_{n[m(r)]}$$

converges as $r \to \infty$ to the continuous function $\hat{G}(v)\exp(v^2\sigma^2/2)$. Now $\{\tilde{Y}_{n[m(r)],j}\}$ are independent for each r and satisfy *uan*, since

$$P(\{|\tilde{Y}_{n[m(r)],j}(\varepsilon_r)| > \varepsilon\}) = P(\{|Z_{n[m(r)],j}| > \varepsilon\}) \tag{6.9}$$

for r sufficiently large that $\varepsilon_r \leqq \varepsilon$. Consequently, by the classical theorem for within-row independent *uan* random variables, we see that $\hat{G}(v)\exp(v^2\sigma^2/2)$ must be the characteristic function of an infinitely divisible law, determined by Q, γ, and σ_0^2, say. Furthermore, we know that, for every $u > 0$ that is a continuity point of Q,

$$\begin{aligned} -Q(u) &= \lim_{r\to\infty} \sum_{j=1}^{k(n[m(r)])} P(\{\tilde{Y}_{n[m(r)],j}(\varepsilon_r) > u\}) \\ &= \lim_{r\to\infty} \sum_{j=1}^{k(n[m(r)])} P(\{Z_{n[m(r)],j} > u\}), \end{aligned}$$

that, for every $u < 0$ that is a continuity point of Q,

$$Q(u) = \lim_{r\to\infty} \sum_{j=1}^{k(n[m(r)])} P(\{\tilde{Y}_{n[m(r)],j}(\varepsilon_r) < u\}) = \lim_{r\to\infty} \sum_{j=1}^{k(n[m(r)])} P(\{Z_{n[m(r)],j} < u\})$$

and that

$$\begin{aligned} \sigma_0^2 &= \lim_{\varepsilon\to 0} \limsup_{r\to\infty} \sum_{j=1}^{k(n[m(r)])} \operatorname{var}[\tilde{Y}_{n[m(r)],j}(\varepsilon_r) I(\{|\tilde{Y}_{n[m(r)],j}(\varepsilon_r)| \leqq \varepsilon\})] \\ &= \lim_{\varepsilon\to 0} \limsup_{r\to\infty} \sum_{j=1}^{k(n[m(r)])} \operatorname{var}[Z_{n[m(r)],j} I(\{\varepsilon_r < |Z_{n[m(r)],j}| \leqq \varepsilon\})]. \end{aligned} \tag{6.10}$$

However, for $0 < \varepsilon_r < \varepsilon$,

$$\operatorname{var}[Z_{n[m(r)],j} I(\{\varepsilon_r < |Z_{n[m(r)],j}| \leqq \varepsilon\})] \leqq E([\hat{Z}_{n[m(r)],j}(\varepsilon)]^2 - [\hat{Z}_{n[m(r)],j}(\varepsilon_r)]^2),$$

so that

$$0 \leqq \sigma_0^2 \leqq \limsup_{r \to 0} \limsup_{r \to \infty} [S(n[m(r)], \varepsilon) - S(n[m(r)], \varepsilon_r)]$$

$$= \limsup_{r \to 0} \left[\limsup_{r \to \infty} S(n[m(r)], \varepsilon) - S' \right]$$

$$\leqq \limsup_{r \to 0} [S'(\varepsilon) - S'] = S' - S' = 0.$$

Consequently, $\hat{G}(v)\exp(v^2\sigma^2/2)$ is the characteristic function of the infinitely divisible law determined by Q, γ, and $\sigma_0^2 = 0$, which shows that $\hat{G}$ is the characteristic function of the infinitely divisible law determined by Q, γ, and σ^2. Hence these quantities are determined uniquely by $\hat{G}$.

From now on, we shall assume that $\varepsilon_0 > 0$ is chosen so that $\pm\varepsilon_0$ are continuity points of Q. Thus what we have shown so far is that for every subsequence $\{n(m)\}$ of $\{n\}$, there exists a sequence $\{\varepsilon_r\}$ of positive numbers converging to zero as $r \to \infty$ and a further subsequence $\{n[m(r)]\}$ such that $\{X_{n[m(r)],j}\}$ satisfies condition (i) for this Q,

$$\lim_{r \to \infty} V(n[m(r)], \varepsilon_r) = \sigma^2,$$

and

$$\sum_{j=1}^{k(n[m(r)])} [\tilde{Y}_{n[m(r)],j}(\varepsilon_r) + \nu_{n[m(r)],j}(\varepsilon_r)] + \sum_{j=1}^{k(m[n(r)])} \mu_{n[m(r)],j}(\varepsilon_0) - A_{n[m(r)]}$$

converges in distribution as $r \to \infty$ to the infinitely divisible law determined by Q, γ, and $\sigma_0^2 = 0$. However, since $\pm\varepsilon_0$ are continuity points of Q, we also know from the classical case that

$$\gamma(\varepsilon_0) = \lim_{r \to \infty} \left[\sum_{j=1}^{k(n[m(r)])} \int_{\{|\tilde{Y}_{n[m(r)],j}(\varepsilon_r)| \leqq \varepsilon_0\}} \tilde{Y}_{n[m(r)],j}(\varepsilon_r) dP + \sum_{j=1}^{k(n[m(r)])} \nu_{n[m(r)],j}(\varepsilon_r) \right.$$
$$\left. + \sum_{j=1}^{k(n[m(r)])} \mu_{n[m(r)],j}(\varepsilon_0) - A_{n[m(r)]} \right]$$

$$= \lim_{r \to \infty} \left[\sum_{j=1}^{k(n[m(r)])} \nu_{n[m(r)],j}(\varepsilon_0) + \sum_{j=1}^{k(n[m(r)])} \mu_{n[m(r)],j}(\varepsilon_0) - A_{n[m(r)]} \right]$$

$$= \lim_{r \to \infty} \left[\sum_{j=1}^{k(n[m(r)])} \mu_{n[m(r)],j}(\varepsilon_0) - A_{n[m(r)]} \right]$$

making use of Lemma 9. At this point we can see that $\{X_{n,j}\}$ must satisfy conditions (i) and (ii) of the main theorem, with (ii) holding for every $\varepsilon_0 > 0$ such that $\pm\varepsilon_0$ are continuity points of Q. Furthermore, applying the corollary to Proposition B, we now see that if $0 < \varepsilon \leqq \varepsilon_0$ with $\pm\varepsilon$ continuity points of Q, then $\sum_{j=1}^{k(n)} [\tilde{Y}_{n,j}(\varepsilon) + \nu_{n,j}(\varepsilon)] + \sum_{j=1}^{k(n)} \mu_{n,j}(\varepsilon_0) - A_n$, which has characteristic function

$$\hat{H}_{n,\varepsilon}(v) = \hat{F}_{n,\varepsilon}(v)\exp[v^2 V(n,\varepsilon)/2],$$

must converge in distribution as $n \to \infty$ to the infinitely divisible law determined by Q_ε, $\gamma(\varepsilon_0)$, and $\sigma_\varepsilon^2 = 0$. This law has characteristic function

$$\exp[iv\gamma(\varepsilon_0) + \psi_\varepsilon(v)] \to \exp[iv\gamma(\varepsilon_0) + \psi_0(v)]$$

as $\varepsilon \to 0$. We also know that $\hat{G}(v) = \exp[iv\gamma(\varepsilon_0) + \psi_0(v)]\exp[-v^2\sigma^2/2]$ and, by (6.3) and (6.4),

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \sup_{|v|\leqq V} |\hat{F}_{n,\varepsilon}(v) - \hat{G}(v)| = 0 \tag{6.11}$$

for every $V > 0$. Suppose that $V(n,\varepsilon) \leqq B$ for $n \geqq 1$ and $0 < \varepsilon \leqq \varepsilon_0$. Then we must have $\sigma^2 \leqq B$ as well. Hence, by the mean value theorem,

$$\begin{aligned}(v^2/2)|V(n,\varepsilon) - \sigma^2| &\leqq \exp(v^2B/2)|\exp[-v^2V(n,\varepsilon)/2] - \exp(-v^2\sigma^2/2)| \\ &\leqq \exp(v^2B/2)\{|\hat{F}_{n,\varepsilon}(v) - \hat{G}(v)| + |\exp[iv\gamma(\varepsilon_0) + \psi_0(v)] - \hat{H}_{n,\varepsilon}(v)|\}/|\hat{H}_{n,\varepsilon}(v)|.\end{aligned}$$

Since

$$\lim_{n\to\infty} |\hat{H}_{n,\varepsilon}(v)| = |\exp[\psi_\varepsilon(v)]| > 0$$

and

$$\lim_{\varepsilon\to 0} |\exp[\psi_\varepsilon(v)]| = |\exp[\psi_0(v)]| > 0,$$

we see that

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} (v^2/2)|V(n,\varepsilon) - \sigma^2| = 0, \tag{6.12}$$

when $\varepsilon \to 0$ through positive values such that $\pm\varepsilon$ are continuity points of Q. Therefore, by Lemma 17, $\{X_{n,j}\}$ satisfies condition (iii) with this value of σ^2.

7. Corollaries of the main theorem

We first consider the case of finite variance, as in Theorem 1.2 (but mistakenly labeled 2.1 on page 175) of Bergström [2].

COROLLARY 1 (Extension of Lindeberg's Theorem). *Let $\{X_{n,j}\}$ be ϕ-mixing with coefficient satisfying Ibragimov's condition, $E(X_{n,j})=0$, $j=1,2,\cdots,k(n)$, and*

$$\sum_{j=1}^{k(n)} \operatorname{var}(X_{n,j}) = 1$$

for $n=1,2,3,\cdots$. Then $\{X_{n,j}\}$ satisfies uan, sjn, and $\sum_{j=1}^{k(n)} X_{n,j}$ converges in distribution as $n\to\infty$ to the $N(0,\sigma^2)$ law provided that

$$\lim_{n\to\infty} \operatorname{var}\left(\sum_{j=1}^{k(n)} X_{n,j}\right) = \sigma^2 \tag{7.1}$$

and Lindeberg's condition is satisfied; that is, for every $\varepsilon>0$,

$$\lim_{n\to\infty} \sum_{j=1}^{k(n)} E([\tilde{X}_{n,j}(\varepsilon)]^2) = 0. \tag{7.2}$$

PROOF. First note that $P(\{|X_{n,j}|>\varepsilon\}) \leqq \varepsilon^{-2}E([\tilde{X}_{n,j}(\varepsilon)]^2) \leqq \varepsilon^{-2}\operatorname{var}(X_{n,j})$ and $\operatorname{var}(\hat{X}_{n,j}(\varepsilon)) \leqq E([\hat{X}_{n,j}(\varepsilon)]^2) \leqq \operatorname{var}(X_{n,j})$. Consequently, $\{X_{n,j}\}$ satisfies *stb* and *svb* in any event. Now if Lindeberg's condition is satisfied, we see that condition (i) of the main theorem is satisfied for $Q\equiv 0$ on $(-\infty,0)\cup(0,\infty)$ which implies *uan* and *sjn*. Furthermore, since $E(X_{n,j})=0$, we have

$$|E(\hat{X}_{n,j}(\varepsilon))| = |E(\tilde{X}_{n,j}(\varepsilon))| \leqq \varepsilon^{-1}E([\tilde{X}_{n,j}(\varepsilon)]^2),$$

so that for every $\varepsilon_0>0$,

$$\lim_{n\to\infty} \sum_{j=1}^{k(n)} \mu_{n,j}(\varepsilon_0) = 0.$$

Finally, since

$$\lim_{n\to\infty} \sum_{j=1}^{k(n)} \operatorname{var}(\tilde{X}_{n,j}(\varepsilon)) = 0$$

for every $\varepsilon>0$, Lemma 16 shows that

$$\lim_{n\to\infty} \operatorname{var}\left(\sum_{j=1}^{k(n)} \tilde{X}_{n,j}(\varepsilon)\right) = 0$$

for every $\varepsilon>0$. Thus we see that for every $\varepsilon>0$,

$$\lim_{n\to\infty} \operatorname{var}\left(\sum_{j=1}^{k(n)} \hat{X}_{n,j}(\varepsilon)\right) = \sigma^2. \tag{7.3}$$

Consequently, $\{X_{n,j}\}$ satisfies the preliminary conditions and conditions (i), (ii), and (iii) of the main theorem, with $A_n = 0$ and $\gamma(\varepsilon_0) = 0$ in (ii), which proves that as $n \to \infty$ $\sum_{j=1}^{k(n)} X_{n,j}$ converges in distribution to the $N(0, \sigma^2)$ law.

Notice that condition (i) being satisfied for $Q \equiv 0$ is equivalent to the sums of the tail probabilities being asymptotically negligible, *stn*; that is,

$$\lim_{n \to \infty} \sum_{j=1}^{k(n)} P(\{|X_{n,j}| > \varepsilon\}) = 0$$

for every $\varepsilon > 0$. Under *stn*, we observe that for ϕ-mixing $\{X_{n,j}\}$ with coefficient satisfying Ibragimov's condition, (7.3) is equivalent to condition (iii) of the main theorem. For if condition (iii) is satisfied we know that $\{\mathrm{var}(\sum_{j=1}^{k(n)} \hat{X}_{n,j}(\varepsilon))\}$ is a bounded sequence for each $\varepsilon > 0$. Since $\mathrm{var}[\sum_{j=1}^{k(n)} \hat{X}_{n,j}(\varepsilon) - \sum_{j=1}^{k(n)} \hat{X}_{n,j}(\varepsilon')]$ approaches zero as $n \to \infty$, by (4.4) and the fact that with $Q \equiv 0$

$$\lim_{n \to \infty} \sum_{j=1}^{k(n)} \mathrm{var}(\hat{X}_{n,j}(\varepsilon) - \hat{X}_{n,j}(\varepsilon')) \leqq \lim_{n \to \infty} \sum_{j=1}^{k(n)} E([\hat{X}_{n,j}(\varepsilon) - \hat{X}_{n,j}(\varepsilon')]^2) = 0,$$

we see that

$$\limsup_{n \to \infty} \mathrm{var}\left(\sum_{j=1}^{k(n)} \hat{X}_{n,j}(\varepsilon)\right) \quad \text{and} \quad \liminf_{n \to \infty} \mathrm{var}\left(\sum_{j=1}^{k(n)} \hat{X}_{n,j}(\varepsilon)\right)$$

do not depend upon ε. Consequently, condition (iii) implies (7.3) in this case. Thus for convergence to the $N(0, \sigma^2)$ distribution we can simplify the main theorem as follows (see theorem 6.5 of Bergström [1]).

COROLLARY 2 (Normal convergence). *Let $\{X_{n,j}\}$ be ϕ-mixing with coefficient fulfilling Ibragimov's condition and suppose $\{X_{n,j}\}$ satisfies stn and svb. Then $\sum_{j=1}^{k(n)} X_{n,j} - A_n$ converges in distribution to the $N(0, \sigma^2)$ law if and only if*

$$\sum_{j=1}^{k(n)} \mu_{n,j}(\varepsilon) - A_n \to 0 \qquad \text{as } n \to \infty \tag{7.4}$$

for some (and hence all) $\varepsilon > 0$ and (7.3), namely

$$\mathrm{var}\left(\sum_{j=1}^{k(n)} \hat{X}_{n,j}(\varepsilon)\right) \to \sigma^2 \qquad \text{as } n \to \infty,$$

for some (and hence all) $\varepsilon > 0$.

For infinitely divisible laws with $\sigma^2 = 0$, for example the non-normal stable laws, we can reduce the question of convergence in certain cases to that of the classical within-row independent situation. One possibility is the following.

COROLLARY 3. *Let $\{X_{n,j}\}$ be ϕ-mixing with coefficient satisfying Ibragimov's condition. Furthermore, suppose $\{X_{n,j}\}$ satisfies uan and sjn. Let $\{Y_{n,j}\}$ be random* *variables, independent for each n, such that* $Y_{n,j}$ *has the same distribution as* $X_{n,j}$. *Suppose that as* $n \to \infty$, $\sum_{j=1}^{k(n)} Y_{n,j} - A_n$ *converges in distribution to the infinitely divisible law determined in the form* (2.4) *by* Q, γ, *and* $\sigma^2 = 0$. *Then* $\sum_{j=1}^{k(n)} X_{n,j} - A_n$ *converges in distribution as* $n \to \infty$ *to this same law.*

PROOF. We can see immediately that $\{X_{n,j}\}$ must satisfy (i) and (ii) of the main theorem for the same Q and $\gamma(\varepsilon_0)$ as $\{Y_{n,j}\}$, using the classical *uan* independent case results. In particular, this shows $\{X_{n,j}\}$ satisfies *stb*. Also, we know that

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \sum_{j=1}^{k(n)} \operatorname{var}(\hat{X}_{n,j}(\varepsilon)) = 0 \tag{7.5}$$

when $\varepsilon \to 0$ through positive values such that $\pm \varepsilon$ are continuity points of Q. This also shows that $\{X_{n,j}\}$ satisfies *svb*, so all preliminary conditions are verified. Finally, using (4.4) to extend Lemma 16 to include iterated limits, we see that

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \operatorname{var}\left(\sum_{j=1}^{k(n)} (\hat{X}_{n,j}(\varepsilon)) \right) = 0, \tag{7.6}$$

again when $\varepsilon \to 0$ through positive values such that $\pm \varepsilon$ are continuity points of Q. Hence $\{X_{n,j}\}$ satisfies (iii) for $\sigma^2 = 0$.

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REFERENCES

1. H. Bergström, *A comparison method for distribution functions of sums of independent and dependent random variables*, Theory Probab. Its Appl. **15** (1970), 430–457.

2. H. Bergström, *On the convergence of sums of random variables in distribution under mixing condition*, Period. Math. Hung. **2** (1972), 173–190.

3. H. Bergström, *Reduction of the limit problem for sums of random variables under a mixing condition*, Proceedings of 4th Conference on Probability Theory, Brasov, 1973, pp. 107–120.

4. P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.

5. B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley, Reading, 1954.

6. I. A. Ibragimov, *Some theorems for stationary processes*, Theory Probab. Its Appl. **7** (1962), 349–382.

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